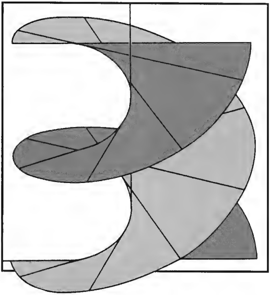
M434 Part IV
Mathematics: A Fourth Level Course

M434
DIFFERENTIAL GEOMETRY



PART IV
CALCULUS ON A SURFACE

The Open University

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Mathematics: A Fourth Level Course

M434 Differential Geometry

Part IV Calculus on a Surface

Prepared for the Course Team
by Bob Margolis

Set book

Barrett O'Neill, *Elementary Differential Geometry*, hardback edition (Academic Press, 1966).

It is essential to have this book; the course is based on it and will not make sense without it.

The set book is referred to as *O'Neill*.

The Open University, Walton Hall, Milton Keynes, MK7 6AA.

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1.2

Contents

Introduction

We now begin the study of surfaces. We shall tackle surfaces using essentially the same approach as we did with curves. We equip the surface being studied witb a frame field and then use the rate of change of the frame field to give geometric information about the shape of the surface.

On curves, we can move only along the curve, so derivatives of tbe Frenet frame were 'ordinary' derivatives with respect to tbe parameter (t or s) used to define the curve. On a surface there are two 'degrees of freedom'; at any point there are two independent directions in which we could move. This corresponds to tbe intuitive idea that surfaces have tangent planes (two-dimensional) rather than tangent lines (one-dimensional) as curves do.

We have already defined most of the tools we shall need. The covariant derivatives connection forms and structural equations give us the methods for discussing rates of change when we can move in arbitrary directions. We need to consider wbat happens to these general ideas when we restrict ourselves to moving about on a surface.

Part II, Sections 5–8.

Note: The section numbering below is tbat of the *text*. There are no text sections corresponding to IV.6 and IV.8 of *O'Neill*.

We begin, in Section 1, by defining what we mean by a surface in $\mathbf{E}^3$. Unlike curves, surfaces will be subsets of $\mathbf{E}^3$. We shall require surfaces to be 'two-dimensional' and 'smooth'. The smoothness will be ensured (as it was for curves) by requiring any functions involved in the definition to be differentiable.

Curves, you will recall, were functions.

As usual, 'differentiable' means differentiable infinitely many times.

To formalize the notion of a surface being two-dimensional, we ask that near eacb point the surface 'looks like' a plane. By 'looks like' we mean that surrounding each point there is a region which is the image of a part of the plane $\mathbf{E}^2$ under a well-behaved (differentiable) mapping.

Unlike curves, we do not ask for a single function to produce the whole of a surface. If you would like an analogy, surfaces are defined in much the same way as children make papier mâché masks: built up of small, overlapping pieces, each of which is a distorted piece of plane. The distortion must be gentle enough to be done by a differentiable function and the pieces must overlap smoothly with no sharp points or ridges.

The *definition* of surface turns out to be a little unwieldy for practical use. We would expect that the set obtained by drawing the graph of a differentiable function

$$f : \mathbf{E}^2 \longrightarrow \mathbf{R},$$

that is, the set

$$M = \{(p_1, p_2, p_3) \in \mathbf{E}^3 : p_3 = f(p_1, p_2)\},$$

to be a surface. We would also expect that the set of points satisfying a relation such as

$$x^2 + y^2 + z^2 = 1$$

(which defines a sphere) to be a surface. The next step after the definition is to prove a theorem that gives a practical test to determine whether a given subset of $\mathbf{E}^3$ is a surface.

The section concludes with some examples.

Section 2 shows how a system of coordinates can be transferred from tbe plane $\mathbf{E}^2$ to a surface and provides some further examples of surfaces. The system of coordinates will be the starting point for defining a frame field adapted to the surface.

Section 3 is fairly technical in nature. We take some of tbe concepts of *Part I*: real-valued functions, tangent vectors and vector fields and show bow they may

be defined on a surface. We also show how the concept of differentiability can be defined for functions whose domain is a surface rather than the real numbers.

Section 4 continues the work of Section 3 by defining what are meant by differential forms and exterior derivative on a surface. We need these definitions because they are essential for the connection forms and Cartan structural equations. As these equations are the analogues of the Frenet formulas, we are expecting to extract geometric information about surfaces from them.

In Section 5, we consider mappings between surfaces. We do so for two reasons. One is, simply, that the possibility of mappings between surfaces exists, so curiosity demands that we investigate! The other is that mappings from a 'standard' surface to a new one can give geometric information about the new surface.

Our 'standard' surface will be a sphere.

Section 6 discusses some properties that surfaces may possess. The three considered: connectedness, compactness and orientability are grouped together because they are topological properties and do not depend directly on notions of differentiability. However, the only one we shall really consider is orientability.

Finally, Section 7 provides a summary.

Note: Sections IV.6 and IV.8 of *O'Neill* do not form part of *M434*. However, Section 6 does provide a link with the integral calculus that you have already studied in other courses. For this reason, there is an appendix to this text which discusses Section IV.6. **The appendix is optional and is not assessed in any way.**

Section IV.8 provides a generalization of the concept of a surface which is important for further work but which is not needed for *M434*.

Study advice

The following represents a possible plan for study weeks.

Week 1 *O'Neill*, Chapter IV, Sections 1–3.

Week 2 *O'Neill*, Chapter IV, Sections 4, 5 and 6.

This leaves 2 study weeks for the work on the first three sections of Chapter V and TMA03. You may well find tackling Sections 4, 5 and 6 in one week a rather heavier than average workload.

1 Surfaces

Read *O'Neill: Chapter IV, Section 1, pages 124–131.*

In this section we begin the programme outlined in the introduction by defining a surface in $\mathbf{E}^3$.

Definition 1.1 The name *coordinate patch* should remind you of the papier mâché analogy used in the introduction. This definition is very much a two-dimensional analogue of the definition of curve.

In intuitive terms, the one–one requirement ensures that the image $\mathbf{x}(D)$ of D cannot cross itself. Since we require $\mathbf{x}$ to be a mapping, it must be differentiable. This ensures that $\mathbf{x}(D)$ is a smoothly distorted piece of plane with no sharp peaks or ridges. We comment on the purpose of regularity below.

Proper patches It is not entirely clear from *O'Neill* why this idea is introduced. The reason is that requiring $\mathbf{x}$ to be a proper patch is enough to ensure that the image set $\mathbf{x}(D)$ is an open set. You may be aware of this from other courses, if not it does not matter.

We want $\mathbf{x}(D)$ to be an open set for Definition 1.2 to work properly.

There is a hint, in Fig. 4.1, as to why 'coordinate' appears in the definition. As usual, we assume that $\mathbf{E}^2$ is provided with Cartesian coordinates. These provide a 'coordinate grid' on the subset D which become a grid of curves on the image set $\mathbf{x}(D)$.

The requirement for regularity of $\mathbf{x}$ ensures that the grid on the image will be a *grid* and not just one set of curves. At each point on D we can place two independent tangent vectors, one along each grid line. Since the images of these tangent vectors must be independent (by regularity), these images will be tangent to the 'grid curves' in the image.

Thus $\mathbf{x}$ provides a 'curvilinear' coordinate system on the image set $\mathbf{x}(D)$, hence 'coordinate patch'.

Definition 1.2 This is the definition that formalizes the papier mâché construction process for surfaces, although it is actually phrased as a test to apply to a set to detect if the set is a surface.

You may find it useful to consider Definition 1.2 from the point of view of a creature living on the surface. The definition asserts that, wherever you are on the surface, you are within a set that looks like a smoothly distorted piece of plane. Because there is a neighbourhood surrounding you that is in the image of a proper patch, you cannot be on the edge of a distorted piece of plane.

Note that a given surface may consist of the image sets of several different proper patches, as the examples in *O'Neill* show. However, where patches overlap they cannot just meet at a point or line. To see how patches overlap, suppose that the point $\mathbf{p}$ of a surface M is in the image sets of both $\mathbf{x}_1$ and $\mathbf{x}_2$. Then there is a neighbourhood N_1 of $\mathbf{p}$ contained in the image of $\mathbf{x}_1$ and a neighbourhood N_2 of $\mathbf{p}$ contained in the image of $\mathbf{x}_2$. The intersection of two neighbourhoods is a neighbourhood. (Because of the way *O'Neill* defines 'neighbourhood' in M the intersection of two neighbourhoods will be the smaller one.)

That is, where the image sets of patches overlap.

Points and lines cannot contain neighbourhoods of any of their points, so patches must overlap in a whole region. This will be important for some of the later, technical theorems.

The sphere Notice how the definition of surface in terms of *proper* patches makes for difficulties in verifying that the sphere is, indeed, a surface.

The first patch defined in *O'Neill* cannot include the equator without breaking the condition that the image is an open set in the surface.

Note that the matrix computed *is* the transpose of the Jacobian as defined earlier in the course. The domain of the patch $\mathbf{x}$ is a subset of $\mathbf{E}^2$ and has dimension 2. The rank of the Jacobian is the same as the dimension of the domain and so $\mathbf{x}$ is regular.

Notation This example introduces some notation that will be used consistently for surfaces. When we are dealing with $\mathbf{E}^2$, or an open subset D of $\mathbf{E}^2$, as the domain of a patch, we shall refer to the coordinate functions as u and v rather than x and y. A consequence is that partial derivatives appear as

$$\frac{\partial}{\partial u} \quad \text{and} \quad \frac{\partial}{\partial v}$$

and the 1-forms appear as

$$du \quad \text{and} \quad dv.$$

This does avoid confusion with the use of **x** for patches. However, it raises the possibility of confusion with **u** and **v** for tangent vectors. You may need to be quite careful in your work to indicate tangent vectors by underlining (where the printed text uses bold type).

Theorem 1.4 The difficulty in checking something as straightforward as a sphere makes something like this theorem highly desirable. You are not expected to follow *O'Neill*'s outline proof because we do not expect you to have met the Implicit Function Theorem. You are expected to be able to apply the test provided by the theorem.

Note that $dg = 0$ at a point p if, and only if,

$$\frac{\partial g}{\partial x}(\mathbf{p}) = \frac{\partial g}{\partial y}(\mathbf{p}) = \frac{\partial g}{\partial z}(\mathbf{p}) = 0.$$

Surfaces of revolution Note that the definition given by *O'Neill* is very restrictive: the profile curve may not cross (or meet) the axis of revolution. That means that, contrary to intuition, the sphere

$$x^2 + y^2 + z^2 = 1$$

is not a surface of revolution!

The definition is made restrictive to avoid the possibility of the 'surface' having a sharp point (if the profile curve meets the axis) or crossing itself (if the curve crosses the axis). The only 'safe' extension to the definition is where the profile curve is symmetric about the axis of revolution and crosses the axis at right angle (avoiding sharp points). This extension is dealt with in Exercise 12, page 133 of *O'Neill*.

Exercise 1.1 *O'Neill*, page 131, Exercise 1. This exercise is intended to reinforce your intuitive understanding of the definition of surface. You should try to describe where and why the definition breaks down in each case.

Exercise 1.2 *O'Neill*, page 132, Exercise 4.

Exercise 1.3 *O'Neill*, page 132, Exercise 5.

Exercise 1.4 *O'Neill*, page 132, Exercise 8.

Exercise 1.5 This exercise concerns the subset M of $\mathbf{E}^3$ defined by

$$M : z = \frac{x^2 - y^2}{4}$$

and the mapping $\mathbf{x} : \mathbf{E}^2 \longrightarrow \mathbf{E}^3$ defined by

$$\mathbf{x}(u, v) = (u + v, u - v, uv).$$

(a) By expressing M in the form $M : g = c$, or otherwise, show that M is a surface in $\mathbf{E}^3$.

(b) Show that $\mathbf{x}(u, v)$ belongs to M for all values of u and v.

(c) Show that **x** is onto M by showing that every point in M can be expressed in the form $\mathbf{x}(u, v)$ for suitable values of u and v.

(d) Show that **x** is a proper patch.

[*Solutions on page 32*]

2 Patch computations

Read *O'Neill: Chapter IV, Section 2, pages 133–140.*

Erratum *O'Neill*, page 136, the calculation of $r^{-2}g(\mathbf{x})$ should read:

$$\begin{aligned} r^{-2}g(\mathbf{x}) &= (\cos v \cos u)^2 + (\cos v \sin u)^2 + \sin^2 v \\ &= \cos^2 v + \sin^2 v = 1. \end{aligned}$$

■

In this section we show how patches provide a system of coordinates on a surface. We also start the process of abandoning patches as being rather restricting to work with and transfer our attention to parametrizations. These, roughly speaking, are patches with the one–one condition dropped.

Parameter curves The idea behind the notion of parameter curves is simple: they are the images of the coordinate grid in the domain of a patch.

There is a tight link between coordinates in $\mathbf{E}^2$, or some open subset D of $\mathbf{E}^2$, and the parameter curves on a surface. Suppose that

$$\mathbf{x} : D \longrightarrow \mathbf{E}^3$$

is one of the patches defining a surface M. We can write the equation of a line through

$$(u_0, v_0)$$

in D, parallel to the x-axis as

$$\alpha(t) = (u_0, v_0) + t(1, 0).$$

The image of α under $\mathbf{x}$ is

$$\mathbf{x}(\alpha(t)).$$

By Definition 2.1, the velocity of $\mathbf{x}(\alpha(t))$ at $t = 0$ is

$$\mathbf{x}_u(u_0, v_0).$$

But, because $\mathbf{x}$ is a mapping, the velocity of $\mathbf{x}(\alpha(t))$ is the image of the velocity of α under the derivative map $\mathbf{x}_*$.

However, the velocity of α is

$$\alpha'(t) = (1, 0)$$

everywhere on α. Hence

$$\mathbf{x}_u(u_0, v_0) = \mathbf{x}_*(1, 0).$$

Using the notation of *O'Neill* for the coordinate functions of $\mathbf{x}$, the Jacobian matrix for $\mathbf{x}_*$ is

$$\begin{pmatrix} \dfrac{\partial x_1}{\partial u}(u_0, v_0) & \dfrac{\partial x_1}{\partial v}(u_0, v_0) \\ \dfrac{\partial x_2}{\partial u}(u_0, v_0) & \dfrac{\partial x_2}{\partial v}(u_0, v_0) \\ \dfrac{\partial x_3}{\partial u}(u_0, v_0) & \dfrac{\partial x_3}{\partial v}(u_0, v_0) \end{pmatrix}.$$

Matrix multiplication by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

then provides the expressions for the partial velocities near the bottom of page 134.

Regularity The partial velocities provide a simple test for regularity of a mapping

$$\mathbf{x} : D \longrightarrow \mathbf{E}^3.$$

The partial velocities form the columns of the Jacobian for $\mathbf{x}_*$. Regularity requires these columns to be linearly independent which is the same as requiring the tangent vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ to be linearly independent.

On the other hand, the cross product of two non-zero vectors is non-zero precisely when the vectors are linearly independent. Thus, $\mathbf{x}$ is regular whenever

$$\mathbf{x}_u \times \mathbf{x}_v \neq 0$$

everywhere on the domain of $\mathbf{x}$.

On page 136, *O'Neill* expresses $\mathbf{x}_u \times \mathbf{x}_v$ as a (formal) determinant. In this expression, the U_i must be interpreted as being composites

$$U_i \circ \mathbf{x} : (u, v) \longmapsto U_i(\mathbf{x}(u, v))$$

so that all the functions appearing in the determinant have the same domain. (If this is not done, the U_i have domain $\mathbf{E}^3$, all other functions have domain part of $\mathbf{E}^2$.)

This is the first example of using the same notation for a function defined on a surface and the composite of such a function with a patch. This abuse of notation will recur from time to time.

Examples The examples of the cylinder, surfaces of revolution and the torus will reappear from time to time in our work on surfaces.

One simple version of the general cylinder is the circular cylinder where the curve C is a circle. For example, we could take

$$C : x^2 + y^2 = r^2$$

with parametrization

$$\alpha(t) = (r \cos t, r \sin t, 0), \quad t \in \mathbf{R}.$$

This gives a corresponding parametrization

$$\mathbf{x}(u, v) = (r \cos u, r \sin u, v)$$

of M.

The partial velocities are

$$\mathbf{x}_u(u, v) = (-r \sin u, r \cos u, 0),$$
$$\mathbf{x}_v(u, v) = (0, 0, 1).$$

Calculating the cross product,

$$\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v) = (r \cos u, r \sin u, 0).$$

Because $\cos u$ and $\sin u$ cannot simultaneously be zero, the cross product is non-zero and the mapping $\mathbf{x}$ is regular, as predicted by the general argument in *O'Neill*.

You might like to note that the *tangent vector* $\mathbf{x}_u \times \mathbf{x}_v$ has the same coordinate functions as α. Therefore it points along the normal to the surface.

Actually, we ought to expect $\mathbf{x}_u \times \mathbf{x}_v$ always to be normal to the surface at $\mathbf{x}(u, v)$. This is because $\mathbf{x}_u$ and $\mathbf{x}_v$ are independent tangent vectors which are tangents (in the usual sense) to the surface and hence define the tangent plane to the surface at $\mathbf{x}(u, v)$. The cross product will be perpendicular to the tangent plane and so normal to the surface. We shall exploit this method of obtaining a normal to a surface later on.

Ruled surfaces This concept is discussed in *O'Neill* only in the exercises for this section. We think it is important enough to mention here.

The basic idea is a generalization of the way cylinders were described by moving a line along a plane curve, keeping the line at right angles to the plane of the curve. The generalization involves two relaxations, allowing general curves in $\mathbf{E}^3$ instead of plane curves and allowing more general rules than 'keep at right angles to the plane of the curve'.

An example of the method of construction may clarify the general procedure.

Consider the curve

$$\alpha(t) = (t, t^2, t^3), \quad t \in \mathbf{R}.$$

Suppose that we draw a line through each point of α in the direction of the tangent to α at that point, in other words, in the direction of α'. We have

$$\alpha'(t) = (1, 2t, 3t^2).$$

To get to a typical point on one of these lines we must go to a point on the curve and then along some multiple of the velocity vector. Thus we should define a mapping $\mathbf{x}$ by

$$\mathbf{x}(u, v) = \alpha(u) + v\alpha'(u).$$

As a check:

$$\mathbf{x}(u, 0) = \alpha(u)$$

and so the curve α lies entirely in the image of $\mathbf{x}$. Also

$$\mathbf{x}_v(u, v) = \alpha'(u)$$

and so the velocities of the v-parameter curves are constant and match the velocity of the curve α.

It is fairly clear from the form of the v-parameter curves that they are straight lines. However, we can check that they are straight lines in the very restricted sense that *O'Neill* defined in Chapter II. The acceleration can be obtained by differentiating $\mathbf{x}_v$ with respect to v:

$$\frac{\partial}{\partial v}\mathbf{x}_v(u, v) = 0.$$

Since the acceleration is zero, the v-parameter curves are, indeed, straight lines.

In this example we can give an explicit form for $\mathbf{x}$:

$$\begin{aligned}\mathbf{x}(u, v) &= (u, u^2, u^3) + v(1, 2u, 3u^2)\\ &= (u + v, u(u + 2v), u^2(u + 3v)).\end{aligned}$$

The partial velocities are given by

$$\begin{aligned}\mathbf{x}_u(u, v) &= (1, 2(u + v), 3u(u + 2v)),\\ \mathbf{x}_v(u, v) &= (1, 2u, 3u^2).\end{aligned}$$

The cross product of the partial velocities is

$$\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v) = (-6u^2v, 6uv, -2v).$$

This product is only zero if

$$u = v = 0.$$

So, if we define D to be $\mathbf{E}^2$ with the origin deleted, then

$$\mathbf{x} : D \longrightarrow \mathbf{E}^3$$

defines a regular mapping.

If we restrict the domain further, to $u, v > 0$, then it is possible to show that $\mathbf{x}$ becomes a proper patch and hence that the image set is a surface.

The general form of the above construction requires two functions: one to take the place of α above and the other to define the direction of each line, taking the place of α' above.

The general parametrization of a ruled surface is of the form

$$\mathbf{x}(u, v) = \beta(u) + v\delta(u).$$

As another example, the surface M defined by

$$M : z = xy$$

This particular surface is called a *saddle surface*.

has a parametrization that shows that it is a ruled surface. If we take the Monge patch suggested by the definition of M

$$\mathbf{x}(u, v) = (u, v, uv),$$

then we can rewrite it as

$$\begin{aligned}\mathbf{x}(u, v) &= (u, 0, 0) + v(0, 1, u) \\ &= \beta(u) + v\delta(u),\end{aligned}$$

where

$$\beta(u) = (u, 0, 0), \quad \text{and} \quad \delta(u) = (0, 1, u).$$

The fact that $\mathbf{x}$ can also be written as

$$\mathbf{x}(u, v) = (0, v, 0) + u(1, 0, v)$$

suggests that both the u and v-parameter curves are straight lines. Thus M is an example of a 'doubly ruled' surface.

We shall come across a number of surfaces which are, in an intuitive sense, curved and yet, like M in the above example, are generated by straight lines.

Exercise 2.1 *O'Neill*, page 140, Exercise 1.

Exercise 2.2 *O'Neill*, page 140, Exercise 2.

Exercise 2.3 *O'Neill*, page 140, Exercise 5.

Exercise 2.4 *O'Neill*, page 141, Exercise 6. Ignore the last part.

Exercise 2.5 *O'Neill*, page 141, Exercise 7.

Exercise 2.6 *O'Neill*, page 142, Exercise 10. Ignore the request for a sketch.

Exercise 2.7 *O'Neill*, page 142, Exercise 11. Ignore the request for a sketch.

Exercise 2.8 *O'Neill*, page 143, Exercise 12. Last part as for the previous exercise.

[*Solutions on page 33*]

3 Functions and tangent vectors

Read *O'Neill: Chapter IV, Section 3, pages 143–149.*

Errata

1 *O'Neill*, page 148, the statement of Lemma 3.8 should read as follows.

If $M : g = c$ is a surface in $\mathbf{E}^3$ and if the gradient vector field

$$\nabla g = \sum \left(\frac{\partial f}{\partial x_i}\right) U_i$$

(considered only at points of M) is never zero, then it is a non-vanishing normal vector field on the entire surface M.

2 *O'Neill*, page 148, the first three lines of the proof of Lemma 3.8 should be deleted. ■

In this section we discuss the differentiability of functions whose domain or codomain is a surface (rather than all of $\mathbf{E}^2$ or $\mathbf{E}^3$) and the notion of a tangent vector being tangent to a surface.

Differentiability When we discussed the calculus of $\mathbf{E}^2$ and $\mathbf{E}^3$ (*Parts I* and *II*) one of the main requirements was that all functions should be differentiable. All definitions of differentiability were, ultimately, based on the differentiability of 'ordinary' functions from $\mathbf{R}$ to $\mathbf{R}$. We shall take a similar approach here. There are two cases to consider: one where the surface is the domain and the other where it is the codomain.

Looking at a particular version of the first case, suppose that we have a surface M and a function

$$f : M \longrightarrow \mathbf{R}.$$

Now M must be defined by one or more (proper) patches. If $\mathbf{x}$ is one such patch, with domain $D \subseteq \mathbf{E}^2$, then the composite

$$f(\mathbf{x}) : (u,v) \longmapsto \mathbf{x}(u,v) \longmapsto f(\mathbf{x}(u,v))$$

is a function from D to $\mathbf{R}$ in the sense discussed in *Part I*. We may, therefore, talk about the composite $f(\mathbf{x})$ being differentiable.

It is reasonable to *define* f to be differentiable if all the composites

$$f(\mathbf{x})$$

are differentiable for all the patches defining M. This is the motivation for the second paragraph of the section on page 143.

In practice it is quite usual to define a function on a surface by giving the composite $f(\mathbf{x})$. For example, suppose that M is the surface defined by the single patch

$$\mathbf{x}(u,v) = (u, v, uv), \quad u, v \in \mathbf{R}.$$

This does define a proper patch.

To define a function f on M we must specify the value that f takes at each point of M. One way to do this is in terms of u and v, since each point of M is of the form $\mathbf{x}(u,v)$. Thus we can define, say,

$$f(\mathbf{x}(u,v)) = u^2 + v^2 + u^2v^2.$$

This definition of f is the same as defining

$$f(\mathbf{p}) = p_1^2 + p_2^2 + p_3^2, \quad \mathbf{p} \in M$$

or

$$f = x^2 + y^2 + z^2, \text{ domain of } f \text{ is } M.$$

The definition of f in terms of the composite makes two things clear: firstly that the domain is M and secondly that the composite is differentiable. This is one of the reasons why failing to distinguish between f and $f(\mathbf{x})$ is not too terrible an abuse.

Note: This definition poses a problem for checking that such functions are differentiable; it is not enough to show that $f(\mathbf{x})$ is differentiable for the patch(es) used to define M. Differentiability must be shown for all conceivable patches in M.

This is a serious drawback and is remedied by the rather technical lemmas in the text. The end result is, reasonably enough, that the check needs to be made only for enough patches to cover the surface. (Thus in the above example, one patch is enough.)

We now turn to the other case: where the surface is the codomain. The situation is the following. We have a surface M, a (proper) patch $\mathbf{x}$ and a function F from, say, $\mathbf{E}^3$ to M

$$D \xrightarrow{\mathbf{x}} M \xleftarrow{F} \mathbf{E}^3.$$

Following the same strategy as in the first case, we need a composite which is an ordinary mapping in the sense of *Part I*. The mapping diagram above gives a clue

as to what to do. We make use of the fact that $\mathbf{x}$ is a proper patch to invert it to give

$$D \xleftarrow{\mathbf{x}^{-1}} M \xleftarrow{F} \mathbf{E}^3$$

or, more conventionally,

$$\mathbf{E}^3 \xrightarrow{F} M \xrightarrow{\mathbf{x}^{-1}} D.$$

This leads to the definition in the third paragraph of the section in *O'Neill*, defining F to he differentiahle if all such composites

$$\mathbf{x}^{-1}(F)$$

are differentiahle as mappings.

The lemmas in *O'Neill* reduce the need to check all patches to checking just enough to cover the surface.

Proofs and results The proofs of 3.1–3.4 are not important, the results are.

The result that we shall make most use of is Lemma 3.1 which describes exactly what curves in a surface look like.

Tangent vectors Suppose that we have a surface M in $\mathbf{E}^3$ and a point $\mathbf{p}$ of M. Since $\mathbf{p}$ is a point of $\mathbf{E}^3$, it has its collection of tangent vectors—its tangent space. Some of these tangent vectors are of particular interest. Firstly those that are tangent to M (in the intuitive sense), secondly those that are normal to M.

Definition 3.5 formalizes the intuitive idea of heing tangent to M hy defining a tangent vector to be a tangent to M if it is the velocity vector of some curve in M. This definition is close to intuition but not very efficient as a practical test. Lemma 3.6 provides the practical test.

Note how the chain rule plays a central role in the proof of Lemma 3.6. There are some notes on the derivation of this form of the chain rule later in the commentary.

There is a passing remark in the proof of Lemma 3.6 that is worth careful note: the partial velocities form a hasis for the tangent plane at each point of $\mathbf{x}(D)$. In a lot of work with surfaces they are the 'natural' choice of basis for the tangent plane hecause they are tightly linked to the definition of the surface via its patches. Calculations are usually hest carried out in terms of the partial velocities. We hope that this general principle will be reinforced hy the examples that you will meet.

There is an application of the chain rule used in the proof of Lemma 3.6 that is worth noting for its multiple uses.

If $\mathbf{x}$ is a parametrization in a surface M and if

$$\alpha(t) = \mathbf{x}(\alpha_1(t), \alpha_2(t))$$

is a curve in the surface M, then

$$\alpha' = \mathbf{x}_u(\alpha_1, \alpha_2)\alpha_1' + \mathbf{x}_v(\alpha_1, \alpha_2)\alpha_2'.$$

This result can he used to find curves that pass through a specified point with a specified velocity at that point.

Vector fields Definition 3.7 is much as you might expect. The reason for paying attention to *normal* vector fields will emerge in *Part V* of the course.

Lemma 3.8 gives one method of generating a normal vector: the gradient. From remarks earlier about partial velocities, it is tempting to use

$$\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v),$$

which is always normal to the surface, to attempt to define a normal vector field. This will not work for reasons that we explore in Section 6 of this text.

The chain rule The situation described in Corollary 3.4 arises frequently in our subsequent work. Although *O'Neill* does cover the corresponding version of the chain rule in an exercise, we think it important enough to deal with here.

The general situation is the following: we have two mappings

$$\mathbf{x} : D_1 \longrightarrow \mathbf{E}^3,$$
$$\mathbf{y} : D_2 \longrightarrow \mathbf{E}^3.$$

We also have two functions

$$\overline{u}, \overline{v} : D_1 \longrightarrow D_2.$$

The problem is to relate the partial velocities of $\mathbf{y}$ with those of $\mathbf{x}$.

We have dealt with exactly this situation in our work on mappings in *Part I*. If we define a mapping F from D_1 to D_2 by

$$F : (u, v) \longmapsto (\overline{u}(u, v), \overline{v}(u, v)),$$

then we have

$$\mathbf{y} = \mathbf{x}(F).$$

It follows, by the composite rule for derivative maps, that

$$\mathbf{y}_* = \mathbf{x}_*(F)F_*.$$

The chain rule that we are aiming for is obtained by expressing this last equation in matrix form. To do so we need to use the coordinate functions of $\mathbf{x}$ and $\mathbf{y}$.

Suppose that

$$\mathbf{x} = (f_1, f_2, f_3),$$
$$\mathbf{y} = (g_1, g_2, g_3).$$

Then the Jacobian matrices for $\mathbf{x}_*$ and $\mathbf{y}_*$ are

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \\ \dfrac{\partial f_3}{\partial u} & \dfrac{\partial f_3}{\partial v} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \dfrac{\partial g_1}{\partial u} & \dfrac{\partial g_1}{\partial v} \\ \dfrac{\partial g_2}{\partial u} & \dfrac{\partial g_2}{\partial v} \\ \dfrac{\partial g_3}{\partial u} & \dfrac{\partial g_3}{\partial v} \end{pmatrix}.$$

(Note that the columns are the partial velocities.) The Jacobian matrix for F_* is

$$\begin{pmatrix} \dfrac{\partial \overline{u}}{\partial u} & \dfrac{\partial \overline{u}}{\partial v} \\ \dfrac{\partial \overline{v}}{\partial u} & \dfrac{\partial \overline{v}}{\partial v} \end{pmatrix}.$$

Applying the composite rule gives

$$\begin{pmatrix} \dfrac{\partial g_1}{\partial u} & \dfrac{\partial g_1}{\partial v} \\ \dfrac{\partial g_2}{\partial u} & \dfrac{\partial g_2}{\partial v} \\ \dfrac{\partial g_3}{\partial u} & \dfrac{\partial g_3}{\partial v} \end{pmatrix} (u, v) = \begin{pmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \\ \dfrac{\partial f_3}{\partial u} & \dfrac{\partial f_3}{\partial v} \end{pmatrix} (\overline{u}, \overline{v}) \begin{pmatrix} \dfrac{\partial \overline{u}}{\partial u} & \dfrac{\partial \overline{u}}{\partial v} \\ \dfrac{\partial \overline{v}}{\partial u} & \dfrac{\partial \overline{v}}{\partial v} \end{pmatrix} (u, v).$$

The first column of the product is

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial u}(\overline{u},\overline{v})\dfrac{\partial \overline{u}}{\partial u} + \dfrac{\partial f_1}{\partial v}(\overline{u},\overline{v})\dfrac{\partial \overline{v}}{\partial u} \\ \dfrac{\partial f_2}{\partial u}(\overline{u},\overline{v})\dfrac{\partial \overline{u}}{\partial u} + \dfrac{\partial f_2}{\partial v}(\overline{u},\overline{v})\dfrac{\partial \overline{v}}{\partial u} \\ \dfrac{\partial f_3}{\partial u}(\overline{u},\overline{v})\dfrac{\partial \overline{u}}{\partial u} + \dfrac{\partial f_3}{\partial v}(\overline{u},\overline{v})\dfrac{\partial \overline{v}}{\partial u} \end{pmatrix}.$$

Since

$$\mathbf{x}_u = \left(\frac{\partial f_1}{\partial u}, \frac{\partial f_2}{\partial u}, \frac{\partial f_3}{\partial u}\right)$$

etc., comparing the first columns of the composite rule equation gives

$$\mathbf{y}_u = \mathbf{x}_u(\overline{u},\overline{v})\frac{\partial \overline{u}}{\partial u} + \mathbf{x}_v(\overline{u},\overline{v})\frac{\partial \overline{v}}{\partial u}.$$

Similarly, the second columns give

$$\mathbf{y}_v = \mathbf{x}_u(\overline{u},\overline{v})\frac{\partial \overline{u}}{\partial v} + \mathbf{x}_v(\overline{u},\overline{v})\frac{\partial \overline{v}}{\partial v}.$$

Note how both partial velocities of $\mathbf{x}$ appear in each of the partial velocities for $\mathbf{y}$.

This form of the chain rule will be used frequently in our subsequent work.

Directional derivatives The final definition in the section is of great importance. It is worth noting that something that was a lemma in $\mathbf{E}^3$ (Lemma 4.6 of *Part I*) has become the *definition* for surfaces. This strategy of using a result as the basis of a more general definition will be used again in *Part VI*.

There are consequences of Definition 3.10 that *O'Neill* deals with in Exercise 4 that we consider of sufficient importance to discuss here. (If you wish you could try Exercise 4 on page 149 before reading what follows.)

We remarked above that it is often useful to work in terms of the partial velocities, since they form a basis for the tangent plane at each point. Thus it would be convenient to have a simple, general method of calculating

$$\mathbf{x}_u[f] \quad \text{and} \quad \mathbf{x}_v[f]$$

for a real-valued function f on a surface.

To be specific, we shall find

$$\mathbf{v}_\mathbf{p}[f],$$

where

$$\mathbf{v} = \mathbf{x}_u(u_0, v_0) \quad \text{at } \mathbf{p} = \mathbf{x}(u_0, v_0).$$

The 'obvious' curve through $\mathbf{x}(u_0, v_0)$ with velocity $\mathbf{x}_u(u_0, v_0)$ is the u-parameter curve

$$\alpha(t) = \mathbf{x}(u_0 + t, v_0).$$

Applying the definition

$$\begin{aligned} \mathbf{v}_\mathbf{p}[f] &= \left(\frac{d}{dt}f(\alpha(t))\right)(0) \\ &= \left(\frac{\partial}{\partial u}f(\alpha(t))\frac{d(u_0+t)}{dt} + \frac{\partial}{\partial v}f(\alpha(t))\frac{dv_0}{dt}\right)(0) \quad \text{(chain rule)} \\ &= \left(\frac{\partial}{\partial u}f(\alpha(t))\right)(0) \\ &= \frac{\partial}{\partial u}(f(\mathbf{x}))(u_0, v_0)). \end{aligned}$$

Generalizing, we have

$$\mathbf{x}_u[f] = \frac{\partial f(\mathbf{x})}{\partial u}.$$

Similar calculations give the corresponding result for $\mathbf{x}_v$:

$$\mathbf{x}_v[f] = \frac{\partial f(\mathbf{x})}{\partial v}.$$

Example Suppose that M is the surface

$$M : z = xy$$

parametrized by

$$\mathbf{x}(u, v) = (u, v, uv).$$

Let f be the function giving the distance of a point on M from the origin, that is, f is the restriction to M of

$$\sqrt{x^2 + y^2 + z^2}.$$

Then

$$\begin{aligned}
\mathbf{x}_u(u, v)[f] &= \frac{\partial f(\mathbf{x})}{\partial u} \\
&= \frac{\partial}{\partial u}(x^2 + y^2 + z^2)(u, v, uv) \\
&= \frac{\partial}{\partial u}\sqrt{u^2 + v^2 + u^2v^2} \\
&= \frac{1}{2}\frac{1}{\sqrt{u^2 + v^2 + u^2v^2}}(2u + 2uv^2) \\
&= \frac{u(1 + v^2)}{\sqrt{u^2 + v^2 + u^2v^2}}.
\end{aligned}$$

Similarly

$$\mathbf{x}_v(u, v)[f] = \frac{v(1 + u^2)}{\sqrt{u^2 + v^2 + u^2v^2}}.$$

You may recall from *Part I* that we observed that directional differentiation (of various kinds) with respect to the natural frame field always reduced to partial differentiation with respect to x, y and z. The results above indicate that, on surfaces, we have a corresponding simplicity if we work with respect to partial velocities. All forms of directional differentiation with respect to the partial velocities will reduce to

$$\frac{\partial}{\partial u} \quad \text{and} \quad \frac{\partial}{\partial v}.$$

The calculations above actually constitute a proof of this observation since covariant differentiation etc. are all based, ultimately, on the definition of

$$\mathbf{v}_{\mathbf{p}}[f].$$

■

In what follows we shall make heavy use of directional differentiation with respect to partial velocities, together with the chain rule.

Exercise 3.1 *O'Neill*, page 149, Exercise 1. The patch referred to is

$$\mathbf{x}(u, v) = (r\cos v\cos u, r\cos v\sin u, r\sin v), \quad -\pi < u < \pi, \quad -\frac{\pi}{2} < v < \frac{\pi}{2}.$$

Exercise 3.2 *O'Neill*, page 149, Exercise 2. The parametrization referred to is

$$\mathbf{x}(u, v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u), \quad 0 < r < R.$$

Exercise 3.3 *O'Neill*, page 150, Exercise 5.

Exercise 3.4 *O'Neill*, page 150, Exercise 8(a) and (b).

Exercise 3.5 Let M be the surface defined by the single patch $\mathbf{x}$, where

$$\mathbf{x}(u,v) = (u, v, uv), \quad (u,v) \in \mathbf{E}^2.$$

Let f be the function defined on M by

$$f(\mathbf{p}) = p_1^2 + p_2^2 - p_3.$$

Find $\mathbf{x}_u[f]$ and $\mathbf{x}_v[f]$.

Exercise 3.6 Let $\mathbf{x}$ and $\mathbf{y}$ be mappings from $\mathbf{E}^2$ to $\mathbf{E}^3$ defined as follows:

$$\mathbf{x}(u,v) = (u, v, uv),$$
$$\mathbf{y}(u,v) = \mathbf{x}(u^2+v^2, u^2-v^2).$$

Use both the chain rule and direct calculation to find $\mathbf{y}_u(u,v)$ and $\mathbf{y}_v(u,v)$ and check that your answers are the same by both methods.

[*Solutions on page 36*]

4 Differential forms on a surface

Read *O'Neill: Chapter IV, Section 4, pages 152–157.*

Erratum *O'Neill*, page 157, in Exercise 5, the first displayed line

for $\mathbf{v_p}[g(f)] = g'(f)\mathbf{v_p}[f]$. read $\mathbf{v_p}[g(f)] = g'(f(\mathbf{p}))\mathbf{v_p}[f]$. ■

In this section we continue the process (begun in the previous section) of extending the definitions of *Part I* to surfaces.

1-forms Since we have defined tangent vectors on a surface, the *Part I* definition of 1-form carries over directly to surfaces.

Examples of 1-forms As with $\mathbf{E}^3$, one way of defining 1-forms is as differentials of real-valued functions. The definition of differential is just as for $\mathbf{E}^3$:

$$df(\mathbf{v_p}) = \mathbf{v_p}[f].$$

The proof that df is a 1-form is exactly the same as the one given in *Part I*.

Two special cases are of particular interest. For a surface M and a patch $\mathbf{x}$ in M, we define two functions as follows:

$$\tilde{u} : \mathbf{x}(u,v) \longmapsto u \quad \text{and} \quad \tilde{v} : \mathbf{x}(u,v) \longmapsto v.$$

These *are* functions because patches are one–one.

Next we calculate the values of $d\tilde{u}$ and $d\tilde{v}$ on $\mathbf{x}_u$ and $\mathbf{x}_v$.

$$\begin{aligned} d\tilde{u}(\mathbf{x}_u) &= \mathbf{x}_u[\tilde{u}] \\ &= \frac{\partial \tilde{u}(\mathbf{x}(u,v))}{\partial u} \\ &= \frac{\partial u}{\partial u} \\ &= 1. \end{aligned} \qquad \begin{aligned} d\tilde{u}(\mathbf{x}_v) &= \mathbf{x}_v[\tilde{u}] \\ &= \frac{\partial \tilde{u}(\mathbf{x}(u,v))}{\partial v} \\ &= \frac{\partial u}{\partial v} \\ &= 0. \end{aligned}$$

$$\begin{aligned} d\tilde{v}(\mathbf{x}_u) &= \mathbf{x}_u[\tilde{v}] \\ &= \frac{\partial \tilde{v}(\mathbf{x}(u,v))}{\partial u} \\ &= \frac{\partial v}{\partial u} \\ &= 0. \end{aligned} \qquad \begin{aligned} d\tilde{v}(\mathbf{x}_v) &= \mathbf{x}_v[\tilde{v}] \\ &= \frac{\partial \tilde{v}(\mathbf{x}(u,v))}{\partial v} \\ &= \frac{\partial v}{\partial v} \\ &= 1. \end{aligned}$$

These may remind you of the results that we obtained for $\mathbf{E}^3$:

$$dx_i(U_j) = \delta_{ij}.$$

From the above results we can calculate the values of $d\tilde{u}$ and $d\tilde{v}$ on any vector $\mathbf{v_p}$ tangent to M. We express $\mathbf{v_p}$ in terms of the partial velocities as, say,

$$\mathbf{v_p} = a\mathbf{x}_u(u,v) + b\mathbf{x}_v(u,v)$$

and then use linearity:

$$\begin{aligned} d\tilde{u}(\mathbf{v_p}) &= d\tilde{u}(a\mathbf{x}_u(u,v) + b\mathbf{x}_v(u,v)) \\ &= ad\tilde{u}(\mathbf{x}_u(u,v)) + bd\tilde{v}(\mathbf{x}_v(u,v)) \\ &= a \times 1 + b \times 0 \\ &= a. \end{aligned}$$

Similar calculations show that

$$d\tilde{v}(\mathbf{v_p}) = b.$$

Thus, just as dx, dy and dz pick out the Euclidean components of a tangent vector in $\mathbf{E}^3$, so $d\tilde{u}$ and $d\tilde{v}$ pick out the $\mathbf{x}_u$ and $\mathbf{x}_v$ components of a vector tangent to a surface.

Abuse of notation In spite of the fact that

$$u \quad \text{and} \quad \tilde{u}$$

are different functions (they have different domains), it is usual to call both of them u. With this abused notation we write

$u : (u,v) \longmapsto u$
$\tilde{u} : \mathbf{x}(u,v) \longmapsto u.$

$$\begin{aligned} du(\mathbf{x}_u) &= 1, \\ du(\mathbf{x}_v) &= 0, \\ dv(\mathbf{x}_u) &= 0, \\ dv(\mathbf{x}_v) &= 1. \end{aligned}$$

2-forms Definition 4.1 is the formal statement of the idea that was mentioned briefly in the commentary on differential forms in *Part I*.

Just for reference, we spell out the linearity conditions.

$$\begin{aligned} \eta(a\mathbf{v}_1 + b\mathbf{v}_2, \mathbf{w}) &= a\eta(\mathbf{v}_1, \mathbf{w}) + b\eta(\mathbf{v}_2, \mathbf{w}), \\ \eta(\mathbf{v}, a\mathbf{w}_1 + b\mathbf{w}_2) &= a\eta(\mathbf{v}, \mathbf{w}_1) + b\eta(\mathbf{v}, \mathbf{w}_2). \end{aligned}$$

Please note carefully the remark just after the proof of Lemma 4.2. This implies that once we know the value of a 2-form on a basis at each point of the surface, then we know all about the 2-form. We shall exploit this to define 2-forms by giving their effect on the basis consisting of the partial velocities. That is we shall often define a 2-form, η say, by giving

$$\eta(\mathbf{x}_u, \mathbf{x}_v).$$

Lemma 4.2 then shows how to calculate the value of η on other tangent vectors. If we have two tangent vectors at a point defined by

$$\mathbf{v} = a\mathbf{x}_u + b\mathbf{x}_v \quad \text{and} \quad \mathbf{w} = c\mathbf{x}_u + d\mathbf{x}_v,$$

then we apply Lemma 4.2 to obtain

$$\eta(\mathbf{v}, \mathbf{w}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \eta(\mathbf{x}_u, \mathbf{x}_v) = (ad - bc)\eta(\mathbf{x}_u, \mathbf{x}_v).$$

Wedge product Definition 4.3 links the formal algebra of wedge products that we did in *Part I* with Definition 4.1.

Example We have defined the 1-forms du and dv above. These enable us to define the 2-form $du \wedge dv$. We can calculate the value of $du \wedge dv$ on $(\mathbf{x}_u, \mathbf{x}_v)$ as follows:

We shall, in future, denote this wedge product by $du\,dv$.

$$\begin{aligned}(du \wedge dv)(\mathbf{x}_u, \mathbf{x}_v) &= du(\mathbf{x}_u)\,dv(\mathbf{x}_v) - du(\mathbf{x}_v)\,dv(\mathbf{x}_u)\\ &= 1 \times 1 - 0 \times 0\\ &= 1.\end{aligned}$$

It follows, from Lemma 4.2, that

$$(du \wedge dv)(a\mathbf{x}_u + b\mathbf{x}_v, c\mathbf{x}_u + d\mathbf{x}_v) = ad - bc. \qquad \blacksquare$$

Exterior derivatives Our eventual aim is to use the connection forms and structural equations to obtain geometric information about surfaces. The structural equations involve exterior derivatives of 1-forms, so these need to be defined for surfaces. As has been the case so far, the definition is given in terms of patches. This is the content of Definition 4.4.

Because patches may overlap, Lemma 4.5 is needed to ensure that there is no conflict in regions of overlap. The reassurance provided by the result is important, the details of the proof are not. You may wish to note that the proof rests on several uses of the chain rule and the fact that

$$\frac{\partial}{\partial u}\frac{\partial f}{\partial v} = \frac{\partial}{\partial v}\frac{\partial f}{\partial u}.$$

The exterior derivative obeys the usual linearity and Leibniz properties. (*O'Neill* relegates the proofs to an exercise, but you are free to assume them.)

Computational methods It is convenient to be able to work with differential forms in terms of du and dv.

Compare this with $\mathbf{E}^3$, where we worked in terms of dx etc.

We now discuss some results which generalize the remarks made in *O'Neill* under Example 4.7.

Firstly, suppose that we are given a 1-form ϕ on a surface M and that $\mathbf{x}$ is a patch in M. Then, ϕ is completely determined (in the image of $\mathbf{x}$) by its values on $\mathbf{x}_u$ and $\mathbf{x}_v$. Define the functions f and g by

$$\begin{aligned}f(\mathbf{x}(u,v)) &= \phi(\mathbf{x}_u(u,v)),\\ g(\mathbf{x}(u,v)) &= \phi(\mathbf{x}_v(u,v)).\end{aligned}$$

Then

$$\begin{aligned}\phi(\mathbf{x}_u(u,v)) &= f(\mathbf{x}(u,v))\,du(\mathbf{x}_u(u,v)) + g(\mathbf{x}(u,v))\,dv(\mathbf{x}_u(u,v)),\\ \phi(\mathbf{x}_v(u,v)) &= f(\mathbf{x}(u,v))\,du(\mathbf{x}_v(u,v)) + g(\mathbf{x}(u,v))\,dv(\mathbf{x}_v(u,v)).\end{aligned}$$

Note: The second term in the first line and the first term in the second line are zero.

Writing these two equations in 'functional' form shows that

$$\phi = f\,du + g\,dv.$$

Thus any 1-form on M can be expressed as a linear combination of du and dv.

If more than one patch is needed to cover M then the linear combination may change from image of patch to image of patch. However, the various lemmas proved in *O'Neill* guarantee that all will agree on any overlaps.

We can now express the exterior derivative of a 1-form

$$\phi = f\,du + g\,dv$$

in terms of the wedge product $du\,dv$. Applying Definition 4.4 to ϕ gives

$$\begin{aligned} d\phi(\mathbf{x}_u, \mathbf{x}_v) &= \frac{\partial}{\partial u}\phi(\mathbf{x}_v) - \frac{\partial}{\partial v}\phi(\mathbf{x}_u) \\ &= \frac{\partial}{\partial u}(f(\mathbf{x})\,du(\mathbf{x}_v) + g(\mathbf{x})\,dv(\mathbf{x}_v)) - \frac{\partial}{\partial v}(f(\mathbf{x})\,du(\mathbf{x}_u) + g(\mathbf{x})\,dv(\mathbf{x}_u)) \\ &= \frac{\partial}{\partial u}(0 + g(\mathbf{x})) - \frac{\partial}{\partial v}(f(\mathbf{x}) + 0) \\ &= \frac{\partial g(\mathbf{x})}{\partial u} - \frac{\partial f(\mathbf{x})}{\partial v}. \end{aligned}$$

However, we showed earlier that

$$(du\,dv)(\mathbf{x}_u, \mathbf{x}_v) = 1,$$

so

$$d\phi(\mathbf{x}_u, \mathbf{x}_v) = \left(\frac{\partial g(\mathbf{x})}{\partial u} - \frac{\partial f(\mathbf{x})}{\partial v}\right)(du\,dv)(\mathbf{x}_u, \mathbf{x}_v).$$

We can write this in functional form as

$$d\phi = \left(\frac{\partial g(\mathbf{x})}{\partial u} - \frac{\partial f(\mathbf{x})}{\partial v}\right) du\,dv.$$

If we calculated $d\phi$ as in *Part I*, we would proceed as follows

$$\begin{aligned} d\phi &= d(f\,du + g\,dv) \\ &= df \wedge du + f\,d(du) + dg \wedge dv + g\,d(dv) \\ &\quad \text{(linearity and two applications of Leibniz)} \\ &= df\,du + dg\,dv \\ &= \left(\frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv\right) du + \left(\frac{\partial g}{\partial u}du + \frac{\partial g}{\partial v}dv\right) dv \\ &= \frac{\partial f}{\partial v}dv\,du + \frac{\partial g}{\partial u}du\,dv \\ &= \left(\frac{\partial g}{\partial u} - \frac{\partial f}{\partial v}\right) du\,dv. \end{aligned}$$

The second set of calculations above involve the same abuse of notation as earlier when we ignored the distinction between u and $\tilde{u}$. Since f and g are defined on M, for any patch $\mathbf{x}$ we automatically have the composites

$$f(\mathbf{x}), g(\mathbf{x}) : D \longrightarrow \mathbf{R},$$

where D is the domain of $\mathbf{x}$. As we have indicated, we are far from scrupulous about distinguishing between f and the composite $f(\mathbf{x})$.

Parametrizations In practice, we often work with parametrizations of a surface rather than collections of patches. For example, earlier we worked with the parametrization

Example 2.6 of Section 2.

$$\mathbf{x}(u, v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u), \quad R > r > 0, \quad (u, v) \in \mathbf{E}^2$$

of the torus of revolution T. Now $\mathbf{x}$ is not a patch, since it is not one–one. However, by using exactly the same *rule* but with a restricted domain, we can create a patch from $\mathbf{x}$ which covers part of T. Thus, choosing a domain D_1

$$D_1 = \{(u, v) \in \mathbf{E}^2 : 0 < u < \pi/2, 0 < v < \pi/2\}$$

gives a patch $\mathbf{x}_1$ which covers part of T.

If we choose a collection of domains, on each of which $\mathbf{x}$ is one–one, then we can produce enough patches to cover T.

Now suppose that we have a function from T to $\mathbf{R}$ defined by

$$f(\mathbf{p}) = p_1^2 + p_2^2 - p_3^2, \quad \mathbf{p} \in T.$$

On *any* of the patches obtained from $\mathbf{x}$, the rule for the composite $f(\mathbf{x})$ will be, after simplification,

$$f(\mathbf{x}) = (R + r\cos u)^2 - r^2\sin^2 u.$$

Thus any calculations using the composites will look the same for all patches and lemmas such as Lemma 4.5 ensure that there is no conflict on overlaps between patches.

We can say, for example, that

$$df = (-2r(R + r\cos u)\sin u - 2r^2 \sin u \cos u)du + 0\,dv$$

working straight from the parametrization rather than with individual patches. All we need to know is that we *could* define a covering set of patches from the parametrization if we wanted to.

Generally speaking it is more convenient to work with a single parametrization for a surface than with a collection of patches. Most of the examples of surfaces in this course will have single parametrizations available.

A single parametrization may not exist.

Exercise 4.1 *O'Neill*, page 157, Exercise 1.

Exercise 4.2 Let M be the surface parametrized by the mapping

$$\mathbf{x}(u, v) = (u, v, uv), \quad (u, v) \in \mathbf{E}^2.$$

(Note that $\mathbf{x}$ is actually a patch.)

Let ϕ and ψ be the 1-forms defined on M as follows. If $\mathbf{v_p}$ is a vector tangent to M at $\mathbf{p} = \mathbf{x}(u, v)$, then

$$\phi(\mathbf{v_p}) = \mathbf{v} \cdot \mathbf{x}_u(u, v),$$
$$\psi(\mathbf{v_p}) = \mathbf{v} \cdot \mathbf{x}_v(u, v).$$

(a) Show that $\mathrm{p} = (-1, 3, -3)$ is in M and that

$$\mathbf{v_p} = (1, 2, 1)_\mathbf{p}$$

is tangent to M.

(b) Find $\phi(\mathbf{v_p})$ and $\psi(\mathbf{v_p})$, where $\mathbf{v_p}$ is as above.

(c) Find the effect of ϕ and ψ on $\mathbf{x}_u$ and $\mathbf{x}_v$.

(d) Assume that

$$\phi = a\,du + b\,dv$$

for some functions a and b. By using the results of the previous part, find a and b.

(e) Express ψ as a linear combination of du and dv.

Exercise 4.3 This exercise continues the previous one and uses the same definitions and notation.

Calculate $\phi \wedge \psi$, expressing your answer as a multiple of $du\,dv$. Hence calculate

$$(\phi \wedge \psi)(\mathbf{x}_u, \mathbf{x}_v).$$

Check this last result by using Definition 4.3 directly.

[*Solutions on page 38*]

5 Mappings of surfaces

Read O'Neill: Chapter IV, Section 5, pages 158–164.

Note: Theorem 5.4 and Example 5.5 will not be of any real importance in this course.

We now turn to the problem of defining mappings between surfaces. Previously a mapping has simply been a differentiable function between Euclidean spaces. There is no difficulty about the idea of a *function* from one surface to another. What requires a little care is the definition of differentiability.

Since two surfaces, M and N say, must each be defined by collections of patches, we transfer the definition of differentiability to the domains of the patches. Fig. 4.29 and Definition 5.1 do this.

This is the third application of 'transfer to domains of patches' to obtain definitions. If we summarize the definitions of differentiability in the three cases, the family resemblance may be clearer.

Firstly, for a function $f : M \longrightarrow \mathbf{R}$, we have

$$D \xrightarrow{\mathbf{x}} M \xrightarrow{f} \mathbf{R}$$

and differentiability of f is defined in terms of differentiability of the composite

$$f\mathbf{x}.$$

Secondly, for a function $F : \mathbf{E}^3 \longrightarrow M$, we have

$$\mathbf{E}^3 \xrightarrow{F} M \xrightarrow{\mathbf{x}^{-1}} D$$

and differentiability is defined in terms of differentiability of the composite

$$\mathbf{x}^{-1}F.$$

The latest definition is a combination of these two. For a function $F : M \longrightarrow N$, we have

$$D \xrightarrow{\mathbf{x}} M \xrightarrow{F} N \xrightarrow{\mathbf{y}^{-1}} E$$

and differentiability is defined in terms of the differentiability of the composite

$$\mathbf{y}^{-1}F\mathbf{x}.$$

Having made the definition, there is the usual remark that it is sufficient to do the check for enough patches to cover both surfaces.

Example 5.2 This example illustrates the fact that we usually work with parametrizations (rather than patches), in the knowledge that we *could* set up patches by using a suitable collection of domains.

The rule, $\mathbf{x}$, given for the geographical patch needs a domain restriction to prevent it being periodic. However, *O'Neill* acknowledges the problem in a comment and then ignores it!

Probably the most important feature of this example is that it shows how mappings between surfaces are usually specified when we are working in $\mathbf{E}^3$. The mapping F is, effectively, defined by

$$F(\mathbf{x}(u, v)) = \mathbf{y}(u, \sin v).$$

What we have done is to define the parameters of the image in terms of the parameters of the point in the domain. That is, F is specified by giving the rule

$$(u, v) \longmapsto (u, \sin v)$$

for the composite

$$\mathbf{y}^{-1}F\mathbf{x}.$$

Uses We shall have one main application of mappings between surfaces. We shall consider a particular type of map between a surface M and the unit sphere. Since the unit sphere is, in some sense, a surface with a 'standard' shape, the amount of distortion needed to map M onto the unit sphere will give a measure of the 'shape' of M

These ideas will be made precise in *Part V*.

Derivative maps As in the case of mappings from $\mathbf{E}^m$ to $\mathbf{E}^n$, we next discuss what mappings do to tangent vectors.

Definition 5.3 is a generalization of the definition of

$$\mathbf{v}_\mathbf{P}[f]$$

for a function f from M to $\mathbf{R}$. It is made in terms of curves in the surfaces.

Examples Whenever a definition like 5.3 is made, it is a good idea to apply it to the partial velocities.

This is because of the central role played by the partial velocities in providing bases for the tangent planes.

The partial velocities are the velocities of the parameter curves. It follows from the definition that

$$F_*(\mathbf{x}_u)$$

is the velocity of the image in N of the u-parameter curve in M, that is,

$$F_*(\mathbf{x}_u(u,v)) = \frac{\partial}{\partial u}F(\mathbf{x}(u,v)).$$

The partial derivative can be calculated by using the chain rule that we have discussed previously.

As a concrete example, we shall pursue Example 5.2 of *O'Neill* a little further. There we have

$$F(\mathbf{x}(u,v)) = \mathbf{y}(u,\sin v).$$

Hence,

$$\begin{aligned}F_*(\mathbf{x}_u(u,v)) &= \frac{\partial}{\partial u}F(\mathbf{x}(u,v))\\ &= \frac{\partial}{\partial u}\mathbf{y}(u,\sin v)\\ &= \mathbf{y}_u(u,\sin v)\frac{\partial u}{\partial u} + \mathbf{y}_v(u,\sin v)\frac{\partial \sin v}{\partial u}\\ &= \mathbf{y}_u(u,\sin v).\end{aligned}$$

Similarly

Note the use of the chain rule.

$$\begin{aligned}F_*(\mathbf{x}_v(u,v)) &= \frac{\partial}{\partial v}F(\mathbf{x}(u,v))\\ &= \frac{\partial}{\partial v}\mathbf{y}(u,\sin v)\\ &= \mathbf{y}_u(u,\sin v)\frac{\partial u}{\partial v} + \mathbf{y}_v(u,\sin v)\frac{\partial \sin v}{\partial v}\\ &= \cos v\,\mathbf{y}_v(u,\sin v).\end{aligned}$$

We can check these results directly. We have

$$F(\mathbf{x}(u,v)) = (\cos u, \sin u, \sin v),$$

$$\frac{\partial}{\partial u}F(\mathbf{x}(u,v)) = (-\sin u, \cos u, 0),$$

$$\frac{\partial}{\partial v}F(\mathbf{x}(u,v)) = (0,0,\cos v).$$

But

$$\mathbf{y}_u(u,v) = (-\sin u, \cos u, 0),$$
$$\mathbf{y}_v(u,v) = (0,0,1).$$

So

$$\mathbf{y}_u(u,\sin v) = (-\sin u, \cos u, 0) = F_*(\mathbf{x}_u(u,v)),$$
$$\mathbf{y}_v(u,\sin v) = (0,0,1),$$

and so

$$\cos v\, \mathbf{y}_v(u,v) = (0,0,\cos v) = F_*(\mathbf{x}_v(u,v)).$$

Warning: The displayed line in the middle of page 161 of *O'Neill* is true but very misleading.

The problem is that the result displayed seems to be in direct contradiction to the result that we obtained in the above example:

$$F_*(x_v(u,v)) = \cos v\, \mathbf{y}_v(u,\sin v).$$

There is no contradiction if you inspect the first paragraph on page 161. The displayed result applies to only the *special case*

$$\mathbf{y}(u,v) = F(\mathbf{x}(u,v)).$$

This is the case where the corresponding map between the domains of $\mathbf{x}$ and $\mathbf{y}$ is the identity:

$$(u,v) \longmapsto (u,v).$$

In the example, we had

$$(u,v) \longmapsto (u,\sin v),$$

Hence the more complicated result.

We strongly advise you to ignore the special case and always work from

$$F_*(\mathbf{x}_u) = \frac{\partial}{\partial u}F(\mathbf{x}) \quad \text{and} \quad F_*(\mathbf{x}_v) = \frac{\partial}{\partial v}F(\mathbf{x}),$$

together with the chain rule, as we did in the last example.

We now know how to transfer points and tangent vectors from one surface to another: points are transferred using mappings, tangent vectors by using the associated derivative maps. We now turn to transfer of differential forms.

Definition 5.6 The transfer of forms defined here is in the ***opposite direction*** to the transfer of points and tangent vectors. The mapping F transfers points from M to N. Similarly, the derivative map transfers tangent vectors on M to tangent vectors on N. However, F^* uses forms on N to define forms on M.

O'Neill talks about 'pullbacks' without explanation. We refer to F^* as a pullback because it transfers information in the opposite direction to F, that is from codomain to domain.

Example We shall develop Example 5.2 from *O'Neill*. We have the following mapping

$$F : \Sigma \longrightarrow C, \quad \text{where } F(\mathbf{x}(u,v)) = \mathbf{y}(u,\sin v).$$

Now, like any surface, C has the 1-forms du and dv defined on it. As a reminder, they satisfy

$$du(a\mathbf{y}_u + b\mathbf{y}_v) = a,$$
$$dv(a\mathbf{y}_u + b\mathbf{y}_v) = b.$$

Let us find the 1-forms

$$F^*(du) \quad \text{and} \quad F^*(dv)$$

on Σ. To know all about a 1-form we need its value on a basis of tangent vectors at each point; the logical basis to use (as ever) consists of the partial velocities, $\mathbf{x}_u$ and $\mathbf{x}_v$.

We shall make use of the results that we have found earlier, namely

$$F_*(\mathbf{x}_u(u,v)) = \mathbf{y}_u(u, \sin v),$$
$$F_*(\mathbf{x}_v(u,v)) = \cos v\, \mathbf{y}_u(u, \sin v).$$

Using Definition 5.6(1),

$$\begin{aligned}(F^*\,du)(\mathbf{x}_u(u,v)) &= du(F_*(\mathbf{x}_u(u,v)))\\ &= du(\mathbf{y}_u(u,\sin v))\\ &= 1.\end{aligned}$$

$$\begin{aligned}(F^*\,du)(\mathbf{x}_v(u,v)) &= du(F_*(\mathbf{x}_v(u,v)))\\ &= du(\cos u\, \mathbf{y}_v(u,\sin v))\\ &= 0.\end{aligned}$$

$$\begin{aligned}(F^*\,dv)(\mathbf{x}_u(u,v)) &= dv(F_*(\mathbf{x}_u(u,v)))\\ &= dv(\mathbf{y}_u(u,\sin v))\\ &= 0.\end{aligned}$$

$$\begin{aligned}(F^*\,dv)(\mathbf{x}_v(u,v)) &= dv(F_*(\mathbf{x}_v(u,v)))\\ &= dv(\cos u\, \mathbf{y}_v(u,\sin v))\\ &= \cos u.\end{aligned}$$

These four results completely define $F^*(du)$ and $F^*(dv)$. If we compare the four results with the values of du and dv on the partial velocities *on* Σ, we see that

$$F^*(du) = du \text{ and } F^*(dv) = \cos u\, dv.$$

To give an example of F_* acting on a 2-form, we shall calculate

$$F^*(du\,dv).$$

We have shown earlier that $(du\,dv)(\mathbf{y}_u, \mathbf{y}_v) = 1$.

Applying Definition 5.6(2) gives

$$\begin{aligned}(F^*(du,dv))(\mathbf{x}_u(u,v),\mathbf{x}_v(u,v)) &= (du\,dv)(F_*(\mathbf{x}_u(u,v)), F_*(\mathbf{x}_v(u,v)))\\ &= (du\,dv)(\mathbf{y}_u(u,\sin v), \cos u\,\mathbf{y}_v(u,\sin v))\\ &= \cos u (du\,dv)(\mathbf{y}_u(u,\sin v), \mathbf{y}_v(u,\sin v))\\ &= \cos u.\end{aligned}$$

As we noted earlier, a 2-form is completely determined by its effect on the pair of partial velocities. Since, on Σ,

$$(du\,dv)(\mathbf{x}_u(u,v), \mathbf{x}_v(u,v)) = 1,$$

we have that

$$(F^*(du,dv))(\mathbf{x}_u(u,v),\mathbf{x}_v(u,v)) = \cos u (du\,dv)(\mathbf{x}_u(u,v),\mathbf{x}_v(u,v)).$$

It follows that we may say

$$F^*(du\,dv) = \cos u\, du\, dv.$$

Theorem 5.7 If pulling back forms is to work properly, then we need the results of this theorem. Only one detail of the proof is really important. The strategy in the only case that *O'Neill* actually proves is as follows. The two forms

$$F^*(d\xi) \quad \text{and} \quad d(F^*(\xi))$$

are evaluated on a basis and the results shown to be equal. As usual, the basis chosen consists of the partial velocities. This is a good general strategy for showing that two 2-forms are equal. Their effects need be checked only for the pair

$$(\mathbf{x}_u, \mathbf{x}_v).$$

Exercise 5.1 (This is a continuation of Example 5.2.) Let Σ be the unit sphere and C be the cylinder parametrized by

$$\mathbf{x}(u,v) = (\cos v \cos u, \cos v \sin u, \sin v),$$
$$\mathbf{y}(u,v) = (\cos u, \sin u, v),$$

respectively, and let F be the mapping from Σ to C defined by

$$F(\mathbf{x}(u,v)) = \mathbf{y}(u, \sin v).$$

Let ϕ_1 and ϕ_2 be the 1-forms defined on C as follows. If $\mathbf{v_p}$ is a tangent vector to C at $\mathbf{p} = \mathbf{y}(u,v)$, then

$$\phi_1(\mathbf{v}) = \mathbf{v} \cdot \mathbf{y}_u(u,v),$$
$$\phi_2(\mathbf{v}) = \mathbf{v} \cdot \mathbf{y}_v(u,v).$$

(a) Calculate the values of ϕ_1 and ϕ_2 on the partial velocities of $\mathbf{y}$. Hence find the values of

$$F^*\phi_1 \text{ and } F^*\phi_2$$

on the partial velocities of $\mathbf{x}$.

(b) Calculate

$$(F^*(\phi_1 \wedge \phi_2))(\mathbf{x}_u(u,v), \mathbf{x}_v(u,v)).$$

Exercise 5.2 Consider the surface M defined by the single patch

$$\mathbf{x}(u,v) = (u, v, uv), \quad (u,v) \in \mathbf{E}^2$$

and the mapping F from M to M defined by

$$F(\mathbf{x}(u,v)) = \mathbf{x}(u+v, u-v).$$

The vector field U is defined on M by

$$U(\mathbf{x}(u,v)) = \frac{\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)}{\|\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)\|}.$$

The 2-form η is defined on M as follows. Suppose that $\mathbf{v_p}$ and $\mathbf{w_p}$ are tangent vectors to M at the point $\mathbf{p} = \mathbf{x}(u,v)$. Then

$$\eta(\mathbf{v}, \mathbf{w}) = \mathbf{v} \times \mathbf{w} \cdot U(\mathbf{p}).$$

Find

$$\eta(\mathbf{x}_u(u,v), \mathbf{x}_v(u,v))$$

and hence find

$$(F^*\eta)(\mathbf{x}_u(u,v), \mathbf{x}_v(u,v)).$$

[*Solutions on page 39*]

6 Topological properties

Read *O'Neill: Chapter IV, Section 7, pages 176–180.*

We need one definition from the (omitted) Section IV.6 of *O'Neill*.

Definition A 2-segment in a surface M is a differentiable mapping from a closed rectangle in $\mathbf{E}^2$ to M. That is, a mapping $\mathbf{x}$

$$\mathbf{x} : D \longrightarrow M,$$

where $D = \{(u,v) \in \mathbf{E}^2 : a \le u \le b,\ c \le v \le d\}$.

The ideas in this section are of great importance in the development of the subject beyond this course. However, we shall make rather limited use of them. The main one that we shall consider is orientability.

Orientability This is a global property: its definition demands that there should exist a 2-form that is non-zero *everywhere* on the surface. However, we can always break up a surface into bits which are orientable by using the patches that define the surface. The image of each patch has to be orientable, for reasons that we shall discuss shortly.

Normal vector fields There is an 'obvious' way to try to construct a normal vector field on a surface: use the cross product of partial velocities. Formally, for a patch $\mathbf{x}$ of the surface M, consider

$$\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v).$$

This is certainly normal to the surface at each point where it is defined. The problem of using this cross product to define a vector field on M arises on the overlap of patches. To see why, we have to recall some results from earlier in *O'Neill*.

If two patches $\mathbf{x}$ and $\mathbf{y}$ overlap in M, then, in the overlap, there are functions $\tilde{u}$ and $\tilde{v}$ such that

$$\mathbf{y}(u,v) = \mathbf{x}(\tilde{u},\tilde{v}).$$

Two applications of the chain rule give

$$\mathbf{y}_u(u,v) = \mathbf{x}_u(\tilde{u},\tilde{v})\frac{\partial \tilde{u}}{\partial u} + \mathbf{x}_v(\tilde{u},\tilde{v})\frac{\partial \tilde{v}}{\partial u},$$
$$\mathbf{y}_v(u,v) = \mathbf{x}_u(\tilde{u},\tilde{v})\frac{\partial \tilde{u}}{\partial v} + \mathbf{x}_v(\tilde{u},\tilde{v})\frac{\partial \tilde{v}}{\partial v}.$$

Careful algebra gives

$$\mathbf{y}_u(u,v) \times \mathbf{y}_v(u,v) = \left(\frac{\partial \tilde{u}}{\partial u}\frac{\partial \tilde{v}}{\partial v} - \frac{\partial \tilde{u}}{\partial v}\frac{\partial \tilde{v}}{\partial u}\right) \mathbf{x}_u(\tilde{u},\tilde{v}) \times \mathbf{x}_v(\tilde{u},\tilde{v}).$$

We cannot be sure that the factor in brackets is 1. Thus the 'definition' using the cross product is possibly inconsistent on overlaps.

What went wrong above is also the clue to the remedy: use the cross product to define a *unit* normal vector field. That is, define $U(\mathbf{x}(u,v))$ by

$$U(\mathbf{x}(u,v)) = \frac{\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)}{\|\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)\|}.$$

The fact that $\mathbf{x}$ is a patch, and hence one–one and regular, ensures that the norm is uniquely defined and never zero. The one–one property ensures that each point $\mathbf{p}$ of M arises from just one pair (u,v) of parameters. (See the exercise on Möbius bands below.)

Since the only difficulty we ran into with overlaps was with the size of the normal, not its direction, this modification solves the problem.

Since every surface can be covered by a collection of patches, we can define a unit normal vector field on the image of each patch by this method. The unit normal cannot vanish and so the image of each patch is orientable, even if the complete surface is not.

This method of constructing unit normal vector fields is of considerable use in *Part V*. It can be extended *carefully* from patches to parametrizations in cases where all choices of parameters for a given point give the same result.

Because of its relative unimportance *for this course*, we have set only one exercise in this section (on Möbius bands).

Exercise 6.1 (This exercise is based on *O'Neill*, pages 180–181, Exercise 7.) Let the curves β and δ be defined by

$$\beta(u) = (\cos u, \sin u, 0),$$
$$\delta(u) = \cos(u/2)\beta(u) + \sin(u/2)U_3.$$

We define the Möbius band M to be the image of the parametrization

$$\mathbf{x}(u,v) = \beta(u) + v\delta(u), \ -\tfrac{1}{2} < v < \tfrac{1}{2}.$$

(a) Calculate the partial velocities and hence show that $\mathbf{x}$ is regular. You may use the results of Exercise 2, Section 2.

(b) The point $\mathbf{p} = (1,0,0)$ lies in M, since

$$\mathbf{p} = \mathbf{x}(0,0).$$

Show that it is possible to find a value u_0 of u with the following properties.

$$\mathbf{p} = \mathbf{x}(u_0, 0),$$
$$\mathbf{x}_u(u_0,0) = \mathbf{x}_u(0,0),$$
$$\mathbf{x}_v(u_0,0) = -\mathbf{x}_v(0,0).$$

Now, suppose that M is orientable so that there exists a non-vanishing normal vector field Z on M.

Define a real-valued function f on M by

$$f(\mathbf{x}(u,v)) = \mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v) \cdot Z(\mathbf{x}(u,v)).$$

(c) Explain why f is differentiable, and hence continuous.

(d) Show that f must vanish somewhere on the parameter curve $v = 0$ and deduce that M is not orientable.

[*Solution on page 40*]

7 Summary

Read *O'Neill: Chapter IV, Section 9, pages 187–188.*

We have now extended most of the ideas from calculus on $\mathbf{E}^n$ to surfaces in $\mathbf{E}^3$. One or two definitions still need to be made, including that of covariant derivative. However, these will be based on the definition of

$$\mathbf{v}_\mathbf{p}[f] \quad \text{and} \quad V[f]$$

for surfaces. (Just as they were in $\mathbf{E}^n$.)

A number of useful principles have been established. The most useful are probably the following.

(a) Where possible, work in terms of the partial velocities.

(b) All forms of directional derivative with respect to the partial velocities reduce to partial differentiation with respect to the parameters.

(c) Showing equality of forms is best tackled by evaluating on the basis consisting of partial velocities.

The partial velocities of patches, and of parametrizations, have played a central role in this text and will continue to do so in the next.

8 Appendix: integration

Read *O'Neill: Chapter IV, Section 6, pages 167–173.*

This appendix is included in case you have been wondering about the link between differential forms and integration.

In the integrals that you have met, for example

M101, M203 and *MST204*

$$\int_0^1 2x\,dx,$$

the expression after the integral sign is what we are now calling a 1-form. More, since

$$2x\,dx = d(x^2),$$

this particular 1-form is the exterior derivative of a function (0-form).

We can express the evaluation of the above integral as follows:

$$\begin{aligned}\int_0^1 2x\,dx &= \int_0^1 d(x^2)\\ &= \left[x^2\right]_0^1\\ &= (x^2)(1) - (x^2)(0)\\ &= 1 - 0 = 1.\end{aligned}$$

This suggests that we could state the Fundamental Theorem of Calculus as follows.

M101 and *M203*

If ϕ is a 1-form on $\mathbf{R}$ and

$$\phi = df$$

for some function $f : \mathbf{R} \longrightarrow \mathbf{R}$, then

$$\int_a^b \phi = f(b) - f(a).$$

The definitions in this section of *O'Neill* seek to extend this definition in two ways. Firstly, to cases where ϕ is a 1-form on a surface. Secondly, to integration of 2-forms.

The first extension is the content of Definition 6.1. In the special case that the 1-form is the exterior derivative of a function we obtain Theorem 6.2 as an extension of the Fundamental Theorem of Calculus.

Definitions 6.3 and 6.4 and Theorem 6.5 provide the extension to 2-forms.

We shall not set any exercises on this section but the following worked example provides some motivation for one of the 2-forms that will be of concern later on.

Example In this example we shall consider the unit sphere Σ, parametrized by

$$\mathbf{x}_u(u,v) = (\cos u\cos v, \cos u\sin v, \sin u), \quad (u,v)\in \mathbf{E}^2.$$

We note that, by defining a suitable collection of subsets of $\mathbf{E}^2$, we could use $\mathbf{x}$ to create a covering of proper patches for Σ.

We now construct a unit normal vector field on Σ by using the partial velocities.

$$\begin{aligned}\mathbf{x}_u &= (-\sin u\cos v, -\sin u\sin v, \cos u),\\ \mathbf{x}_v &= (-\cos u\sin v, \cos u\cos v, 0),\\ \mathbf{x}_u\times\mathbf{x}_v &= (-\cos^2 u\cos v, -\cos^2 u\sin v, -\cos u\sin u(\cos^2 v + \sin^2 v))\\ &= -\cos u(\cos u\cos v, \cos u\sin v, \sin u).\\ \|\mathbf{x}_u\times\mathbf{x}_v\|^2 &= \cos^2 u(\cos^2 u\cos^2 v + \cos^2 u\sin^2 v + \sin^2 u)\\ &= \cos^2 u.\end{aligned}$$

From the last two lines we can see that either of

$$\pm\frac{\mathbf{x}_u \times \mathbf{x}_v}{\cos u}$$

would define a unit normal. We choose to define

$$\begin{aligned} U(\mathbf{x}(u,v)) &= \frac{\mathbf{x}_u \times \mathbf{x}_v}{\cos u} \\ &= (-\cos u \cos v, -\cos u \sin v, -\sin u). \end{aligned}$$

Strictly, we should check that all choices of u and v for a particular point lead to a unique value of U. We shall not do so, although all is well.

Next, we use U to define a 2-form on Σ. If $\mathbf{v}$ and $\mathbf{w}$ are tangent vectors at $\mathbf{p} = \mathbf{x}(u,v)$, then we define the 2-form η by

$$\eta(\mathbf{v},\mathbf{w}) = \mathbf{v} \times \mathbf{w} \cdot U.$$

It is not difficult to check that η is, indeed, a 2-form.

If we evaluate η on the partial velocities, then we can express η in terms of $du\,dv$.

$$\begin{aligned} \eta(\mathbf{x}_u, \mathbf{x}_v) &= \mathbf{x}_u \times \mathbf{x}_v \cdot U \\ &= \cos u\, U \cdot U \quad \text{(from the definition of } U\text{)} \\ &= \cos u. \end{aligned}$$

The last step uses the fact that U is a *unit* vector field.

Comparing this result with

$$(du\,dv)(\mathbf{x}_u, \mathbf{x}_v) = 1,$$

we have

$$\eta = \cos u\, du\, dv.$$

Let us now try integrating this 2-form over the 'northern' hemisphere, H defined by

$$0 \le u \le \frac{\pi}{2} \quad \text{and} \quad 0 \le v \le 2\pi.$$

Theorem 6.5 can be applied provided that we can find a 1-form, ϕ, such that

$$\eta = d\phi.$$

Suppose that $\phi = a\,du + b\,dv$. Then, from first principles,

We could also have used Example 4.7 from *O'Neill*.

$$\begin{aligned} d\phi &= \left(\frac{\partial a}{\partial u}du + \frac{\partial a}{\partial v}dv\right) du + \left(\frac{\partial b}{\partial u}du + \frac{\partial b}{\partial v}dv\right) dv \\ &= \left(\frac{\partial b}{\partial u} - \frac{\partial a}{\partial v}\right) du\, dv. \end{aligned}$$

If $d\phi = \eta$, then we want

$$\frac{\partial b}{\partial u} - \frac{\partial a}{\partial v} = \cos u.$$

If we choose

$$b = \sin u \quad \text{and} \quad a = 0$$

then the requirement is satisfied.

Since the boundary of the northern hemisphere is the equator α defined by $u = 0$, $0 \le v \le 2\pi$, Theorem 6.5 gives

$$\int_H \eta = \int_\alpha \phi.$$

We now apply Theorem 6.2 to ϕ. Since we have $u = 0$ on the equator,

$$\begin{aligned}\int_\alpha \phi &= \int_0^{2\pi} \cos(0)\, dv \\ &= \int_0^{2\pi} dv \\ &= [v]_0^{2\pi} \\ &= 2\pi.\end{aligned}$$

■

The significance of this result is that it is half the surface area of the sphere. (You may know the formula $4\pi r^2$ for the area of a sphere of radius r).

The process of constructing η can be generalized to any patch on any surface. The resulting 2-form is usually known as the *area* 2-form. The example above indicates why: integrating the area 2-form over a region gives the area of that region.

There is a geometric connection with areas that makes the result above intuitively reasonable. Two tangent vectors $\mathbf{v}$ and $\mathbf{w}$ at a point define a parallelogram. The magnitude of their cross product is

$$\|\mathbf{v}\|\,\|\mathbf{w}\| \sin\theta,$$

where θ is the angle between them. This is also the (signed) area of the parallelogram defined by the vectors. The effect of the definition

$$\eta(\mathbf{v}, \mathbf{w}) = \mathbf{v} \times \mathbf{w} \cdot U$$

is that $\eta(\mathbf{v}, \mathbf{w})$ gives the area of the parallelogram. Thus η is closely connected with ideas of area.

We shall not pursue integration of forms further. However, the brief discussion above does indicate the links both with previous work and with some that is to follow.

Solutions to the exercises

Solution 1.1

(a) **Cone** The problem here is the point (vertex) of the cone. There is no well-defined tangent plane at the point and so the point cannot be in the image of a patch (let alone a proper patch).

(b) **Closed disc** The points on the rim of the disc cannot meet the requirement of having a neighbourhood contained in the image of a proper patch. They fail precisely because they are right on the edge.

(c) **Folded plane** To understand exactly what goes wrong here, we need to decide what the set is. The condition

$$xy = 0$$

forces $x = 0$ or $y = 0$. Taken together with the condition

$$x \geq 0, \quad y \geq 0$$

this means that we have half the plane $x = 0$ and half the plane $y = 0$. These meet along a sharp edge: the z-axis. Thus M fails the 'smoothness' test.

Solution 1.2

For each mapping we test for being one–one and regular. (Each *is* a mapping because all the coordinate functions are differentiable.)

(a) Suppose that

$$\mathbf{x}(u_1, v_1) = \mathbf{x}(u_2, v_2).$$

Then

$$\begin{aligned}
&(u_1, u_1v_1, v_1) = (u_2, u_2v_2, v_2)\\
&\Rightarrow u_1 = u_2,\ u_1v_1 = u_2v_2,\ v_1 = v_2\\
&\quad \text{(comparing coordinates)}\\
&\Rightarrow (u_1, v_1) = (u_2, v_2).
\end{aligned}$$

Hence $\mathbf{x}$ is one–one.

The Jacobian matrix for $\mathbf{x}_*$ is

$$\begin{pmatrix} 1 & 0 \\ v & u \\ 0 & 1 \end{pmatrix}.$$

Looking at the zeros and ones, we can see that the columns are linearly independent and so the matrix has rank 2. Thus $\mathbf{x}$ is regular.

This completes the proof that $\mathbf{x}$ is a patch.

Note that $\mathbf{x}$ is actually a proper patch since the inverse

$$(p_1, p_2, p_3) \longmapsto (p_1, p_3)$$

has continuous coordinate functions.

(b) First we tackle the one–one property.

$$\begin{aligned}
&\mathbf{x}(u_1, v_1) = \mathbf{x}(u_2, v_2)\\
&\Rightarrow u_1^2 = u_2^2,\ u_1^3 = u_2^3,\ v_1 = v_2\\
&\Rightarrow u_1^3 = u_2^3\\
&\Rightarrow u_1 = u_2\\
&\quad (u \longmapsto u^3 \text{ is one–one})\\
&\Rightarrow (u_1, v_1) = (u_2, v_2).
\end{aligned}$$

Thus $\mathbf{x}$ is one–one.

The Jacobian is

$$\begin{pmatrix} 2u & 0 \\ 3u^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

The rows are *not* independent in the case $u = 0$ because the matrix reduces to

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $\mathbf{x}$ is not regular everywhere on its domain and is not a patch.

(c) We adopt the same approach as above.

$$\begin{aligned}
&\mathbf{x}(u_1, v_1) = \mathbf{x}(u_2, v_2)\\
&\Rightarrow (u_1, u_1^2, v_1 + v_1^3) = (u_2, u_2^2, v_2 + v_2^3)\\
&\Rightarrow u_1 = u_2,\ u_1^2 = u_2^2,\ v_1 + v_1^3 = v_2 + v_2^3.
\end{aligned}$$

From the above we can deduce directly that $u_1 = u_2$. Getting at the vs is slightly more work.

$$\begin{aligned}
&v_1 + v_1^3 = v_2 + v_2^3\\
&\Rightarrow v_1 - v_2 + v_1^3 - v_2^3 = 0\\
&\Rightarrow (v_1 - v_2) + (v_1 - v_2)(v_1^2 + v_1v_2 + v_2^2) = 0\\
&\Rightarrow (v_1 - v_2)(1 + v_1^2 + v_1v_2 + v_2^2) = 0.
\end{aligned}$$

Now, completing the square shows that

$$\begin{aligned}
1 + v_1^2 + v_1v_2 + v_2^2 &= 1 + \left(v_1 + \frac{v_2}{2}\right)^2 + \frac{3v_2^2}{4}\\
&\geq 1.
\end{aligned}$$

Hence we must have

$$\begin{aligned}
&v_1 - v_2 = 0\\
&v_1 = v_2.
\end{aligned}$$

That completes the proof that $\mathbf{x}$ is one–one.

The Jacobian is

$$\begin{pmatrix} 1 & 0 \\ 2u & 0 \\ 0 & 1 + 3v^2 \end{pmatrix}.$$

The first and last rows show that the matrix has independent columns and so has rank 2. Thus $\mathbf{x}$ is a patch.

Note: It is true, although not obvious, that $\mathbf{x}$ is a proper patch. The inverse of

$$v \longmapsto v + v^3$$

is continuous even though it is difficult to express explicitly.

(d) Because the domain is the whole of $\mathbf{E}^2$, the appearance of $\cos 2\pi u$ prevents $\mathbf{x}$ from being one–one. A counterexample provides a complete proof.

$$\begin{aligned}
\mathbf{x}(0, 0) &= (\cos 0, \sin 0, 0)\\
&= (1, 0, 0)\\
&= (\cos 2\pi, \sin 0, 0)\\
&= \mathbf{x}(1, 0).
\end{aligned}$$

We would not expect different values of v to be any use in constructing a counterexample because of the appearance of v in the third coordinate function of $\mathbf{x}$.

Since $\mathbf{x}$ is not one–one, it is not a patch. Calculation of the Jacobian would show, however, that $\mathbf{x}$ is regular.

Solution 1.3

We apply the $dg \neq 0$ test in each case.

(a) Here we define

$$g = (x^2 + y^2)^2 + 3z^2$$

and then M becomes

$$M : g = 1$$

and we can apply the test.

$$dg = 2(x^2 + y^2)2x\,dx + 2(x^2 + y^2)2y\,dy + 6z\,dz.$$

The only way we can have $dg = 0$ is when

$$x = y = z = 0.$$

However, $x = y = z = 0$ do not satisfy $g = 1$ and so dg cannot be zero on M. Hence M is a surface.

(b) We consider

$$g = z(z-2) + xy$$

We have

$$dg = y\,dx + x\,dy + 2(z-1)\,dz.$$

We have $dg = 0$ when

$$x = y = 0, z = 1.$$

Now, M will be a surface provided that these values do *not* satisfy $g = c$. We have

$$g(0, 0, 1) = -1.$$

So, M is a surface for all values of c *except* $c = -1$.

Solution 1.4

We first show that $\mathbf{x}$ is one–one.

$$\mathbf{x}(u_1, v_1) = \mathbf{x}(u_2, v_2)$$
$$\Rightarrow u_1^2 = u_2^2,\ u_1v_1 = u_2v_2,\ v_1^2 = v_2^2.$$

Now, because of the restriction of the domain to *positive* values of u and v, we can deduce $u_1 = u_2$ and $v_1 = v_2$ from the equality of their squares. (Note how the property of being one–one depends on the domain as well as the formulas.)

We also have that $\mathbf{x}$ is regular because the Jacobian is

$$\begin{pmatrix} 2u & 0 \\ v & u \\ 0 & 2v \end{pmatrix}.$$

The only values that would give this a rank of less than 2 are $u = v = 0$ which are excluded from the domain.

The formula for the inverse is obtained from

$$(u^2, uv, v^2) = (p_1, p_2, p_3)$$

to give

$$\mathbf{x}^{-1} : (p_1, p_2, p_3) \longmapsto (\sqrt{p_1}, \sqrt{p_3}).$$

Or

$$\mathbf{x}^{-1} = (\sqrt{x}, \sqrt{z}).$$

From this, $\mathbf{x}^{-1}$ is continuous and $\mathbf{x}$ is a proper patch.

Solution 1.5

(a) If we rewrite the definition of M as

$$M : \frac{x^2 - y^2}{4} - z = 0,$$

We can see that this is

$$M : g = c,$$

where

$$g = \frac{x^2 - y^2}{4} - z \quad \text{and} \quad c = 0.$$

Now

$$dg = \tfrac{1}{2}x\,dx - \tfrac{1}{2}y\,dy - dz$$

which cannot be zero because of the third term. Hence M is a surface.

(b) We must show that $\mathbf{x}(u, v)$ satisfies the equation for M. Since a point $\mathbf{p}$ lies on M if, and only if, it satisfies

$$\left(\frac{x^2 - y^2)}{4} - z\right)(\mathbf{p}) = 0,$$

we evaluate that function on $\mathbf{x}(u, v)$. We have

$$\begin{aligned}
&\left(\frac{x^2 - y^2}{4} - z\right)(\mathbf{x}(u, v)) \\
&= \left(\frac{x^2 - y^2}{4} - z\right)(u + v, u - v, uv) \\
&= \tfrac{1}{4}\left((u + v)^2 - (u - v)^2\right) - uv \\
&= \tfrac{1}{4}(u^2 + 2uv + v^2 - u^2 + 2uv - v^2) - uv \\
&= \tfrac{1}{4}4uv - uv \\
&= 0.
\end{aligned}$$

Thus $\mathbf{x}(u, v)$ lies in M for all u and v.

(c) To show that $\mathbf{x}$ is onto M we must show that

$$\mathbf{x}(u, v) = (p_1, p_2, p_3)$$

can be solved for u and v for any point

$$\mathbf{p} = (p_1, p_2, p_3)$$

on M.

We require

$$\begin{aligned}
u + v &= p_1 \\
u - v &= p_2 \\
uv &= p_3.
\end{aligned}$$

The first two equations can be solved to give

$$\begin{aligned}
u &= \tfrac{1}{2}(p_1 + p_2) \\
v &= \tfrac{1}{2}(p_1 - p_2).
\end{aligned}$$

It remains to show that these values also satisfy the third equation.

$$\begin{aligned}
uv &= \tfrac{1}{4}(p_1 + p_2)(p_1 - p_2) \\
&= \tfrac{1}{4}(p_1^2 - p_2^2) \\
&= p_3.
\end{aligned}$$

The last step follows because $\mathbf{p}$ lies on M and so satisfies the equation for M.

(d) The fact that we were able to solve the equations above *uniquely* shows that $\mathbf{x}$ is one–one. The inverse

$$(p_1, p_2, p_2) \longmapsto (\tfrac{1}{2}(p_1 + p_2), \tfrac{1}{2}(p_1 - p_2))$$

is continuous.

It remains to show that $\mathbf{x}$ is regular. The Jacobian is

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ v & u \end{pmatrix}.$$

The ones and minus ones show that the columns are independent and the matrix has rank 2.

This completes the proof that $\mathbf{x}$ is a proper patch.

Solution 2.1

(a) The technique from Example 2.5 of *O'Neill* applies directly. We have a parametrization of C as

$$\alpha(t) = (t, \cosh t, 0)$$

so

$$g(u) = u, \quad h(u) = \cosh u,$$

leading to the parametrization

$$\mathbf{x}(u, v) = (u, \cosh u \cos v, \cosh u \sin v).$$

(b) We must first parametrize C. Now, C is a circle, centre $(0, 0, 2)$, radius 1 in the yz-plane. This suggests

$$\alpha(t) = (0, \cos t, 2 + \sin t).$$

We are rotating about the y-axis, so the function corresponding to g in Example 2.5 is the y-coordinate function. This leads to the parametrization

$$\mathbf{x}(u, v) = ((2 + \sin u) \cos v, \cos u, (2 + \sin u) \sin v).$$

Note: In any of these parametrizations the roles of $\cos v$ and $\sin v$ may be interchanged if you wish.

(c) The way in which C is defined suggests the following parametrization of C.

$$\alpha(t) = (t, 0, t^2).$$

Since we are rotating about the z-axis, the g function is the third coordinate function of α. This gives

$$\mathbf{x}(u, v) = (u \cos v, u \sin v, u^2).$$

Solution 2.2

We know that the norm of the cross product is given by

$$\|\mathbf{x}_u \times \mathbf{x}_v\|^2 = \|\mathbf{x}_u\|^2 \|\mathbf{x}_v\|^2 \sin^2 \theta,$$

where θ is the angle between the partial velocities.

However, we also know that

$$\begin{aligned} E &= \mathbf{x}_u \cdot \mathbf{x}_u \\ &= \|\mathbf{x}_u\|^2; \\ F &= \mathbf{x}_u \cdot \mathbf{x}_v \\ &= \|\mathbf{x}_u\| \, \|\mathbf{x}_v\| \cos \theta; \\ G &= \mathbf{x}_v \cdot \mathbf{x}_v \\ &= \|\mathbf{x}_v\|^2. \end{aligned}$$

Hence

$$\begin{aligned} \|\mathbf{x}_u \times \mathbf{x}_v\|^2 &= \|\mathbf{x}_u\|^2 \|\mathbf{x}_v\|^2 \sin^2 \theta \\ &= \|\mathbf{x}_u\|^2 \|\mathbf{x}_v\|^2 (1 - \cos^2 \theta) \\ &= EG - F^2. \end{aligned}$$

Since $\mathbf{x}$ is regular if, and only if,

$$\mathbf{x}_u \times \mathbf{x}_v \neq 0,$$

it follows that $\mathbf{x}$ is regular if, and only if,

$$EG - F^2 \neq 0.$$

Solution 2.3

We start by calculating the partial velocities.

$$\begin{aligned} \mathbf{x}_u(u, v) &= 0 + v\delta'(u) \\ &= v\delta'(u); \\ \mathbf{x}_v(u, v) &= 0 + \delta(u) \\ &= \delta(u). \end{aligned}$$

The mapping $\mathbf{x}$ will be regular if, and only if, the cross product is non-zero. We have

$$\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v) = v\delta'(u) \times \delta(u).$$

Thus x is regular wherever both v and $\delta' \times \delta$ are non-zero, as claimed.

Solution 2.4

Using the same approach as in the previous solution we have:

$$\begin{aligned} \mathbf{x}_u(u, v) &= \beta'(u), \\ \mathbf{x}_v(u, v) &= \mathbf{q}, \\ \mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v) &= \beta'(u) \times \mathbf{q}. \end{aligned}$$

The result follows.

Solution 2.5

(a) We start by tackling regularity and then deal with the one–one property.

$$\begin{aligned} \mathbf{x}_u(u, v) &= (\cos v, \sin v, 0), \\ \mathbf{x}_v(u, v) &= (-u \sin v, u \cos v, b), \\ \mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v) &= (b \sin v, -b \cos v, u). \end{aligned}$$

We are given $b \neq 0$ and $\cos v$ and $\sin v$ cannot be zero simultaneously. Thus

$$\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v) \neq 0$$

and $\mathbf{x}$ is regular everywhere.

Now we prove that $\mathbf{x}$ is one–one. Suppose that

$$(u_1 \cos v_1, u_1 \sin v_1, bv_1) = (u_2 \cos v_2, u_2 \sin v_2, bv_2).$$

Since $b \neq 0$, comparing the last coordinates gives

$$v_1 = v_2.$$

Since at least one of $\cos v_1$ or $\sin v_1$ is non-zero, we can deduce that

$$u_1 \cos v_1 = u_2 \cos v_2 = u_2 \cos v_1$$

or

$$u_1 \sin v_1 = u_2 \sin v_2 = u_2 \sin v_1.$$

In either case,

$$u_1 = u_2.$$

Thus $\mathbf{x}$ is one–one and regular, hence a patch.

(b) Fixing $v = v_0$ gives

$$\begin{aligned} \mathbf{x}(u, v_0) &= (u \cos v_0, u \sin v_0, bv_0) \\ &= (0, 0, bv_0) + u(\cos v_0, \sin v_0, 0), \end{aligned}$$

which indicates that the u-parameter curves are straight lines.

Fixing $u = u_0$ gives

$$\mathbf{x}(u_0, v) = (u_0 \cos v, u_0 \sin v, bv),$$

which defines a (circular) helix. (This gives some indication of where the name *helicoid* comes from.)

(c) Provided that we avoid places where $\cos v = 0$ or $u = 0$, we can write

$$\frac{u \sin v}{u \cos v} = \tan v.$$

This suggests an implicit equation of the form

$$\frac{y}{x} = \tan \frac{z}{b}.$$

We can restore the excluded values by rewriting this as

$$y = x \tan \frac{z}{b}.$$

Thus

$$H : g = c,$$

where

$$g = y - x \tan \frac{z}{b}, \quad c = 0.$$

Solution 2.6

We can actually use a single proof to show that all three are surfaces. Each is given in the form

$g = c$

and we have

$dg = \frac{2x}{a^2}\,dx + \frac{2y}{b^2}\,dy \pm \frac{2z}{c^2}\,dz.$

In each case, $dg = 0$ if, and only if

$x = y = z = 0.$

In none of the cases does the origin satisfy

$g = c.$

It follows that all three are surfaces in $\mathbf{E}^3$.

Now we tackle the parametrizations and images for each surface in turn. We assume that, in all cases,

$a, b, c > 0.$

To show that $\mathbf{x}$ is a parametrization, we must show that $\mathbf{x}$ is regular and that each point of the image lies in the surface. To shorten some of the long expressions, we shall omit the parameters (u, v).

(a) We apply the cross product test for regularity.

$$\mathbf{x}_u = (-a\sin u\cos v, -b\sin u\sin v, c\cos u),$$
$$\mathbf{x}_v = (-a\cos u\sin v, b\cos u\cos v, 0),$$
$$\mathbf{x}_u \times \mathbf{x}_v = (-bc\cos^2 u\cos v, -ac\cos^2 u\sin v, -ab\sin u\cos u)$$

For the values of u given,

$\cos u \neq 0$

and $\cos v$ and $\sin v$ cannot be zero simultaneously, so the cross product is non-zero. Hence $\mathbf{x}$ is regular on D.

Also, $\mathbf{x}(D)$ lies in M because

$$\begin{aligned}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)(\mathbf{x}) &= \cos^2 u\cos^2 v + \cos^2 u\sin^2 v \\ &\quad + \sin^2 u \\ &= \cos^2 u(\cos^2 v + \sin^2 v) \\ &\quad + \sin^2 u \\ &= \cos^2 u + \sin^2 u \\ &= 1.\end{aligned}$$

The restriction

$-\frac{\pi}{2} < u < \frac{\pi}{2}$

means that $\sin u$ lies in the range

$-1 < \sin u < 1$

and $\cos u$ in the range

$0 < \cos u \leq 1.$

Since v is unrestricted, we have

$-a \leq a\cos u\cos v \leq a,$

$-b \leq b\cos u\sin v \leq b$

$-c < c\sin u < c.$

Thus the image $\mathbf{x}(D)$ is all of M *except* the two points $(0, 0, \pm c)$.

(b) Here

$$\mathbf{x}_u = (a\sinh u\cos v, b\sinh u\sin v, c\cosh u),$$
$$\mathbf{x}_v = (-a\cosh u\sin v, b\cosh u\cos v, 0),$$
$$\begin{aligned}\mathbf{x}_u \times \mathbf{x}_v &= (-bc\cosh^2 u\cos v, -ac\cosh^2 u\sin v, \\ &\qquad ab\cosh u\sinh u) \\ &= \cosh u(-bc\cosh u\cos v, -ac\cosh u\sin v, \\ &\qquad ab\sinh u).\end{aligned}$$

Now, $\cosh u$ is never zero and $\cos v$ and $\sin v$ cannot be simultaneously zero so the first two components cannot both be zero and $\mathbf{x}$ is regular.

Now

$$\begin{aligned}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)(\mathbf{x}) &= \cosh^2 u\cos^2 v + \cosh^2 \sin^2 v - \sinh^2 u \\ &= \cosh^2 u(\cos^2 v + \sin^2 v) - \sinh^2 u \\ &= \cosh^2 u - \sinh^2 u \\ &= 1.\end{aligned}$$

Hence $\mathbf{x}(\mathbf{R}) \subseteq M$ and $\mathbf{x}$ is a parametrization of M.

The image $\mathbf{x}(D)$ is the whole of M.

(c) Using the same approach as above:

$$\mathbf{x}_u = (a\cosh u\cos v, b\cosh u\sin v, c\sinh u),$$
$$\mathbf{x}_v = (-a\sinh u\sin v, b\sinh u\cos v, 0),$$
$$\begin{aligned}\mathbf{x}_u \times \mathbf{x}_v &= (-bc\sinh^2 u\cos v, -ac\sinh^2 u\sin v, ab\sinh u\cosh u \\ &= \sinh u(-bc\sinh u\cos v, -ac\sinh u\sin v, ab\cosh u).\end{aligned}$$

The restriction $u \neq 0$ ensures that $\sinh u$ is non-zero, $\cosh u$ is never zero and $\cos v$ and $\sin v$ cannot be simultaneously zero. Hence $\mathbf{x}$ is regular.

We have

$$\begin{aligned}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)(\mathbf{x}) &= \sinh^2 u\cos^2 v + \sinh^2 u\sin^2 v - \cosh^2 \\ &= \sinh^2 u - \cosh^2 u \\ &= -1.\end{aligned}$$

So $\mathbf{x}(D) \subseteq M$ and $\mathbf{x}$ is a parametrization of M.

The restriction $u \neq 0$ means that

$c\cosh u > c$

and so

$\frac{z^2}{c^2} > 1.$

This excludes the two points

$(0, 0, \pm c),$

both of which satisfy the equation for M. Thus the image is all of M except these two points.

Solution 2.7

(a) We can use the $dg \neq 0$ test if we rewrite the definition of M as

$M : g = 0,$

where

$g = \frac{x^2}{a^2} + \frac{y^2}{b^2} - z.$

We have

$dg = \frac{2x}{a^2}\,dx + \frac{2y}{b^2}\,dy - dz \neq 0.$

Hence M is a surface.

Next, we show that $\mathbf{x}$ is a parametrization.

$$\mathbf{x}_u = (a\cos v, b\sin v, 2u),$$
$$\mathbf{x}_v = (-au\sin v, bu\cos v, 0),$$
$$\mathbf{x}_u \times \mathbf{x}_v = (-2bu^2\cos v, -2au^2\sin v, abu).$$

Since $u > 0$ and $\cos v$ and $\sin v$ cannot both be zero, $\mathbf{x}$ is regular.

Since

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)(\mathbf{x}) = u^2\cos^2 v + u^2\sin^2 v$$
$$= u^2$$
$$= z(\mathbf{x}(u,v)),$$

the image set of $\mathbf{x}$ is a subset of M. Hence $\mathbf{x}$ is a parametrization of M.

The restriction $u > 0$ excludes the point $(0,0,0)$ which lies in M.

(b) Points on the parameter curve $u = u_0$ have the form

$$(au_0\cos v, bu_0\sin v, u_0^2)$$

and satisfy

$$\frac{x^2}{a^2u_0^2}+\frac{y^2}{b^2u_0^2} = 1, \quad z = u_0^2.$$

These curves are, therefore, ellipses.

Points on the parameter curve $v = v_0$ have the form

$$\mathbf{x}(u,v_0) = (au\cos v_0, bu\sin v_0, u^2)$$
$$= u(a\cos v_0, b\sin v_0, 0) + u^2(0,0,1).$$

The parameter curve, therefore, lies in the plane defined by the two vectors

$$(a\cos v_0, b\sin v_0, 0) \quad \text{and} \quad (0,0,1).$$

If we introduce a 'w-axis' in the direction of

$$(a\cos v_0, b\sin v_0, 0),$$

the equation of the parameter curve becomes

$$z = w^2.$$

This shows that the parameter curve is a parabola.

The nature of the parameter curves indicates why the surface is called an elliptic paraboloid.

Solution 2.8

(a) We must show that $\mathbf{x}$ is a proper patch and onto M. We have

$$\mathbf{x}_u = (a,b,4v),$$
$$\mathbf{x}_v = (a,-b,4u),$$
$$\mathbf{x}_u\times\mathbf{x}_v = (4b(u+v), 4a(v-u), -2ab).$$

Since, for the definition of M to make sense, $a, b \neq 0$ the cross product is non-zero and $\mathbf{x}$ is regular.

Since

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)(\mathbf{x}) = (u+v)^2-(u-v)^2$$
$$= u^2+2uv+v^2-u^2+2uv-v^2$$
$$= 4uv$$
$$= z(\mathbf{x}),$$

the image set of $\mathbf{x}$ is a subset of M and $\mathbf{x}$ is a parametrization.

Finally, suppose that

$$\mathbf{p} = (p_1,p_2,p_3)$$

belongs to M. Then we can solve

$$p_1 = a(u+v)$$
$$p_2 = b(u-v)$$

to obtain

$$u = \frac{1}{2}\left(\frac{p_1}{a}+\frac{p_2}{b}\right)$$
$$v = \frac{1}{2}\left(\frac{p_1}{a}-\frac{p_2}{b}\right).$$

We have expressed $\mathbf{p}$ in the form

$$\mathbf{p} = \mathbf{x}(u,v)$$

for suitable u and v and so $\mathbf{x}$ is onto M.

(b) The two ruled forms are

$$\mathbf{x}(u,v) = (au,bu,0)+v(a,-b,4u)$$
$$\mathbf{x}(u,v) = (av,-bv,0)+u(a,b,4v).$$

(c) Writing the parameter curves in the forms

$$\mathbf{x}(u_0,v) = (au_0,bu_0,0)+v(a,-b,4u_0),$$
$$\mathbf{x}(u,v_0) = (av_0,-bv_0,0)+u(a,b,4v_0),$$

shows that they are straight lines.

Solution 3.1

We calculate the expressions $f(\mathbf{x})$ in each case.

(a) Here we have $f = x^2+y^2$, so

$$f(\mathbf{x}(u,v)) = r^2\cos^2 v\cos^2 u + r^2\cos^2 v\sin^2 u$$
$$= r^2\cos^2 v(\cos^2 u+\sin^2 u)$$
$$= r^2\cos^2 v.$$

(b) This time

$$f(\mathbf{x}(u,v)) = (r\cos v\cos u - r\cos v\sin u)^2 + r^2\sin^2 v$$
$$= r^2\cos^2 v(\cos^2 u+\sin^2 u - 2\cos u\sin u)$$
$$+ r^2\sin^2 v$$
$$= r^2\cos^2 v(1-\sin 2u) + r^2\sin^2 v$$
$$= r^2 - r^2\cos^2 v\sin 2u$$
$$= r^2(1-\cos^2 v\sin 2u).$$

Solution 3.2

(a) We calculate $\mathbf{x}(\alpha(t))$.

$$\mathbf{x}(\alpha(t)) = ((R+r\cos t)\cos t, (R+r\cos t)\sin t, r\sin t).$$

Thus

$$\alpha_1(t) = (R+r\cos t)\cos t,$$
$$\alpha_2(t) = (R+r\cos t)\sin t,$$
$$\alpha_3(t) = r\sin t.$$

(b) Since the coordinate functions involve $\sin t$ and $\cos t$, they will repeat after 2π, that is,

$$\alpha(t+2\pi) = \alpha(t),$$

for all values of t. The only question that arises is whether they can repeat earlier.

So, suppose that

$$\alpha(t) = \alpha(t'),$$

for some values of t and t'. Now, looking at the third coordinate function of α, we can deduce that

$$\sin t = \sin t'.$$

If $\sin t = \sin t'$ is not zero, then the second coordinate function shows that

$$R+r\cos t = R+r\cos t'$$

and hence

$$\cos t = \cos t'.$$

On the other hand, if

$$\sin t = \sin t' = 0,$$

then

$$\cos t = \pm 1, \ \cos t' = \pm 1.$$

Since $r < R$, the expressions

$$R+r\cos t, \ R+r\cos t'$$

are both positive and so the equation

$$R + r\cos t = R + r\cos t'$$

forces $\cos t$ and $\cos t'$ to have the same sign. Thus

$$\cos t = \cos t' = \pm 1.$$

Summing up,

$$\sin t = \sin t'$$

and, regardless of whether or not this common value is zero,

$$\cos t = \cos t'.$$

These equations are satisfied only if

$$t' = t + 2n\pi,$$

for some integer n.

Hence α is periodic with period 2π.

Solution 3.3

We have to apply the basic test: that $\mathbf{v}$ is tangent to M if, and only if, we can write $\mathbf{v}$ as a linear combination of the partial velocities at its point of application.

We begin by defining a parametrization; the appropriate one here is the Monge patch.

$$\mathbf{x}(u,v) = (u, v, f(u,v)).$$

(a) First we calculate the partial velocities.

$$\mathbf{x}_u(u,v) = \left(1, 0, \frac{\partial f}{\partial u}(u,v)\right),$$

$$\mathbf{x}_v(u,v) = \left(0, 1, \frac{\partial f}{\partial v}(u,v)\right).$$

Now, suppose that

$$\mathbf{v}_\mathbf{p} = (v_1, v_2, v_3)_{(p_1,p_2,p_3)}.$$

Then

$$\mathbf{p} = \mathbf{x}(p_1, p_2).$$

We also note that, for the Monge patch, partial differentiation with respect to u and v is the same as with respect to x and y.

Thus, $\mathbf{v}$ is tangent to M if, and only if, $\mathbf{v}$ can be written as a linear combination of $\mathbf{x}_u(p_1,p_2)$ and $\mathbf{x}_v(p_1,p_2)$.

Inspecting the expressions for the partial velocities above, this condition reduces to

$$(v_1, v_2, v_3) = v_1\mathbf{x}_u(p_1,p_2) + v_2\mathbf{x}_v(p_1,p_2).$$

Comparing the third coordinates and using the observation about partial derivatives, we arrive at

$$v_3 = \frac{\partial f}{\partial x}(p_1,p_2)v_1 + \frac{\partial f}{\partial y}(p_1,p_2)v_2.$$

(b) We use the observation made in the last section that

$$\mathbf{x}_u(u,v) \times \mathbf{x}_v(u,v)$$

is normal to the surface at $\mathbf{x}(u,v)$.

Now $\mathbf{v}$ is tangent to M at $\mathbf{x}(u,v)$ if, and only if, it is perpendicular to the normal at $\mathbf{x}(u,v)$. This is true if, and only if,

$$\mathbf{v}\cdot\mathbf{x}_u(u,v)\times\mathbf{x}_v(u,v) = 0.$$

Solution 3.4

(a) The easiest approach is to use the chain rule because it automatically expresses α' in terms of the partial velocities.

$$\begin{aligned}\alpha'(t) &= \mathbf{x}_u(\sqrt{2}t, e^t)\frac{d\sqrt{2}t}{dt} + \mathbf{x}_v(\sqrt{2}t, e^t)\frac{de^t}{dt}\\ &= \sqrt{2}\mathbf{x}_u(\sqrt{2}t, e^t) + e^t\mathbf{x}_v(\sqrt{2}t, e^t).\end{aligned}$$

Note: We have not needed to find explicit expressions for the partial velocities yet.

(b) If we use the result just found, we have

$$\alpha'\cdot\mathbf{x}_u = \sqrt{2}\mathbf{x}_u\cdot\mathbf{x}_u + e^t\mathbf{x}_v\cdot\mathbf{x}_u,$$

$$\alpha'\cdot\mathbf{x}_v = \sqrt{2}\mathbf{x}_u\cdot\mathbf{x}_v + e^t\mathbf{x}_v\cdot\mathbf{x}_v.$$

We also have

$$\begin{aligned}\mathbf{x}_u(u,v) &= v(-\sin u, \cos u, 0),\\ \mathbf{x}_v(u,v) &= (\cos u, \sin u, 1),\\ \mathbf{x}_u(u,v)\cdot\mathbf{x}_u(u,v) &= v^2,\\ \mathbf{x}_u(u,v)\cdot\mathbf{x}_v(u,v) &= 0,\\ \mathbf{x}_v(u,v)\cdot\mathbf{x}_v(u,v) &= 2.\\ \|\mathbf{x}_u(u,v)\| &= |v|,\\ \|\mathbf{x}_v(u,v)\| &= \sqrt{2}.\end{aligned}$$

Putting these results together, we obtain

$$\begin{aligned}\alpha'\cdot\mathbf{x}_u/\|\mathbf{x}_u\| &= \sqrt{2}(e^t)^2/e^t\\ &= \sqrt{2}e^t,\\ \alpha'\cdot\mathbf{x}_v/\|x_v\| &= e^t \times 2/\sqrt{2}\\ &= \sqrt{2}e^t.\end{aligned}$$

Thus α' makes equal angles with the unit vectors

$$\frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} \text{ and } \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}$$

and the result follows.

Solution 3.5

In order to apply the result

$$\mathbf{x}_u[f] = \frac{\partial f(\mathbf{x})}{\partial u}$$

we need to find $f(\mathbf{x}(u,v))$.

Now

$$\begin{aligned}f(\mathbf{x}(u,v)) &= f(u, v, uv)\\ &= u^2 + v^2 - uv.\end{aligned}$$

Thus

$$\begin{aligned}\mathbf{x}_u[f] &= \frac{\partial f(\mathbf{x}(u,v))}{\partial u}\\ &= \frac{\partial}{\partial u}(u^2 + v^2 - uv)\\ &= 2u - v.\end{aligned}$$

Similarly

$$\begin{aligned}\mathbf{x}_v[f] &= \frac{\partial f(\mathbf{x}(u,v))}{\partial v}\\ &= \frac{\partial}{\partial v}(u^2 + v^2 - uv)\\ &= 2v - u.\end{aligned}$$

Solution 3.6

First we calculate $\mathbf{y}(u,v)$ in terms of u and v.

$$\begin{aligned}\mathbf{y}(u,v) &= \mathbf{x}(u^2 + v^2, u^2 - v^2)\\ &= (u^2 + v^2, u^2 - v^2, (u^2 + v^2)(u^2 - v^2))\\ &= (u^2 + v^2, u^2 - v^2, u^4 - v^4).\end{aligned}$$

The four partial velocities are

$$\begin{aligned}\mathbf{x}_u(u,v) &= (1, 0, v),\\ \mathbf{x}_v(u,v) &= (0, 1, u),\\ \mathbf{y}_u(u,v) &= (2u, 2u, 4u^3),\\ \mathbf{y}_v(u,v) &= (2v, -2v, -4v^3).\end{aligned}$$

In the notation used above for deriving the chain rule, we have

$$\overline{u}(u,v) = u^2 + v^2,$$
$$\overline{v}(u,v) = u^2 - v^2.$$

Hence

$$\frac{\partial \overline{u}}{\partial u} = 2u,$$
$$\frac{\partial \overline{u}}{\partial v} = 2v,$$
$$\frac{\partial \overline{v}}{\partial u} = 2u,$$
$$\frac{\partial \overline{v}}{\partial v} = -2v.$$

Using the chain rule

$$\mathbf{y}_u(u,v) = \mathbf{x}_u(\tilde{u},\tilde{v})\frac{\partial \tilde{u}}{\partial u} + \mathbf{x}_v(\tilde{u},\tilde{v})\frac{\partial \tilde{v}}{\partial u}$$

to calculate $\mathbf{y}_u$ gives

$$\begin{aligned}\mathbf{y}_u(u,v) &= \mathbf{x}_u(u^2+v^2, u^2-v^2) \times 2u \\ &\quad + \mathbf{x}_v(u^2+v^2, u^2-v_2) \times 2u \\ &= 2u(1,0,u^2-v^2) + 2u(0,1,u^2+v^2) \\ &= (2u, 2u, 4u^3).\end{aligned}$$

A similar calculation gives

$$\begin{aligned}\mathbf{y}_v(u,v) &= \mathbf{x}_u(u^2+v^2, u^2-v^2) \times 2v \\ &\quad + \mathbf{x}_v(u^2+v^2, u^2-v_2) \times (-2v) \\ &= 2v(1,0,u^2-v^2) - 2v(0,1,u^2+v^2) \\ &= (2v, -2v, -4v^3).\end{aligned}$$

These correspond to the results obtained by direct calculation.

Solution 4.1

We can show equality of 2-forms by evaluating them on a pair of linearly independent vectors at each point of the surface. For each patch $\mathbf{x}$ in the surface, we have

$$\begin{aligned}(\phi \wedge \psi)(\mathbf{x}_u, \mathbf{x}_v) &= \phi(\mathbf{x}_u)\psi(\mathbf{x}_v) - \phi(\mathbf{x}_v)\psi(\mathbf{x}_u) \\ &= -(\psi(\mathbf{x}_u)\phi(\mathbf{x}_v) - \psi(\mathbf{x}_v)\phi(\mathbf{x}_u)) \\ &= -(\psi \wedge \phi)(\mathbf{x}_u, \mathbf{x}_v).\end{aligned}$$

The result follows.

Note: We have chosen to use the partial velocities, you could have chosen any pair $(\mathbf{v}, \mathbf{w})$ of linearly independent tangent vectors at each point. The details of the argument are exactly the same.

Since we have

$$\phi \wedge \phi = -\phi \wedge \phi,$$

it follows that $\phi \wedge \phi = 0$.

Solution 4.2

We shall need the partial velocities, so we calculate them first.

$$\mathbf{x}_u = (1,0,v),$$
$$\mathbf{x}_v = (0,1,u).$$

(a) By inspection of the first two coordinates we have

$$\mathbf{p} = (-1,3,-1) = \mathbf{x}(-1,3).$$

Hence $\mathbf{p} \in M$.

We have

$$\mathbf{x}_u(-1,3) = (1,0,3),$$
$$\mathbf{x}_v(-1,3) = (0,1,-1).$$

Since

$$\mathbf{v}_\mathbf{p} = \mathbf{x}_u(-1,3) + 2\mathbf{x}_v(-1,3),$$

it follows that $\mathbf{v}_\mathbf{p}$ is tangent to M.

(b) Applying the definition of ϕ:

$$\begin{aligned}\phi(\mathbf{v}_\mathbf{p}) &= \mathbf{v} \cdot \mathbf{x}_u(-1,3) \\ &= (1,2,1)\cdot(1,0,3) \\ &= 4.\end{aligned}$$

$$\begin{aligned}\psi(\mathbf{v}_\mathbf{p}) &= \mathbf{v} \cdot \mathbf{x}_v(-1,3) \\ &= (1,2,1)\cdot(0,1,-1) \\ &= 1.\end{aligned}$$

(c) Again, using the definition,

$$\begin{aligned}\phi(\mathbf{x}_u) &= \mathbf{x}_u(u,v)\cdot\mathbf{x}_u(u,v) \\ &= (1,0,v)\cdot(1,0,v) \\ &= 1+v^2.\end{aligned}$$

$$\begin{aligned}\phi(\mathbf{x}_v) &= \mathbf{x}_v(u,v)\cdot\mathbf{x}_u(u,v) \\ &= (0,1,u)\cdot(1,0,v) \\ &= uv.\end{aligned}$$

$$\begin{aligned}\psi(\mathbf{x}_u) &= \mathbf{x}_u(u,v)\cdot\mathbf{x}_v(u,v) \\ &= (1,0,v)\cdot(0,1,u) \\ &= uv.\end{aligned}$$

$$\begin{aligned}\psi(\mathbf{x}_v) &= \mathbf{x}_v(u,v)\cdot\mathbf{x}_v(u,v) \\ &= (0,1,u)\cdot(0,1,u) \\ &= 1+u^2.\end{aligned}$$

(d) We have

$$\begin{aligned}\phi(\mathbf{x}_u) &= (a\,du + b\,dv)(\mathbf{x}_u) \\ &= a\,du(\mathbf{x}_u) + b\,dv(\mathbf{x}_u) \\ &= a \times 1 + b \times 0 \\ &= a.\end{aligned}$$

$$\begin{aligned}\phi(\mathbf{x}_v) &= (a\,du + b\,dv)(\mathbf{x}_v) \\ &= a\,du(\mathbf{x}_v) + b\,dv(\mathbf{x}_v) \\ &= a \times 0 + b \times 1 \\ &= b.\end{aligned}$$

Comparing these to the results from the previous part gives

$$a = (1+v^2),\ b = uv.$$

Thus

$$\phi = (1+v^2)du + uv\,dv.$$

(e) If we assume that

$$\psi = a\,du + b\,dv,$$

then exactly the same method shows that

$$a = uv,\ b = (1+u^2).$$

Hence

$$\psi = uv\,du + (1+u^2)dv.$$

Solution 4.3

Applying the results obtained above:

$$\begin{aligned}\phi \wedge \psi &= ((1+v^2)du + uv\,dv) \wedge (uv\,du + (1+u^2)dv) \\ &= (1+v^2)(1+u^2)du\,dv + (uv)(uv)dv\,du \\ &\quad \text{(since } du\,du = dv\,dv = 0\text{)} \\ &= (1+v^2)(1+u^2)du\,dv - (uv)(uv)du\,dv \\ &\quad \text{(using alternation)} \\ &= (1+u^2+v^2)du\,dv.\end{aligned}$$

Since

$$(du\,dv)(\mathbf{x}_u, \mathbf{x}_v) = 1,$$

we have

$$\begin{aligned}(\phi \wedge \psi)(\mathbf{x}_u, \mathbf{x}_v) &= (1 + u^2 + v^2)(du\,dv)(\mathbf{x}_u, \mathbf{x}_v)\\ &= 1 + u^2 + v^2.\end{aligned}$$

Calculating directly from Definition 4.3 gives

$$\begin{aligned}(\phi \wedge \psi)(\mathbf{x}_u, \mathbf{x}_v) &= \phi(\mathbf{x}_u)\psi(\mathbf{x}_v) - \phi(\mathbf{x}_v)\psi(\mathbf{x}_u)\\ &= (1 + v^2)(1 + u^2) - (uv)(uv)\\ &= 1 + u^2 + v^2,\end{aligned}$$

as before.

Solution 5.1

From the earlier work on this example, we know that

$$F_*(\mathbf{x}_u(u, v)) = \mathbf{y}_u(u, \sin v),$$
$$F_*(\mathbf{x}_v(u, v)) = \cos u\, \mathbf{y}_v(u, \sin v).$$

We shall also need the partial velocities of $\mathbf{y}$.

$$\mathbf{y}_u(u, v) = (-\sin u, \cos u, 0),$$
$$\mathbf{y}_v(u, v) = (0, 0, 1).$$

(a) From the above, we have

$$\begin{aligned}\phi_1(\mathbf{y}_u(u, v)) &= \mathbf{y}_u(u, v) \cdot \mathbf{y}_u(u, v)\\ &= 1;\\ \phi_1(\mathbf{y}_v(u, v)) &= \mathbf{y}_v(u, v) \cdot \mathbf{y}_u(u, v)\\ &= 0;\\ \phi_2(\mathbf{y}_u(u, v)) &= \mathbf{y}_u(u, v) \cdot \mathbf{y}_v(u, v)\\ &= 0;\\ \phi_2(\mathbf{y}_v(u, v)) &= \mathbf{y}_v(u, v) \cdot \mathbf{y}_v(u, v)\\ &= 1.\end{aligned}$$

It follows that

$$\begin{aligned}(F^*\phi_1)(\mathbf{x}_u(u, v)) &= \phi_1(F_*(\mathbf{x}_u(u, v)))\\ &= \phi_1(\mathbf{y}_u(u, \sin v))\\ &= 1.\\ (F^*\phi_1)(\mathbf{x}_v(u, v)) &= \phi_1(F_*(\mathbf{x}_v(u, v)))\\ &= \phi_1(\cos v\, \mathbf{y}_v(u, \sin v))\\ &= \cos v\, \phi_1(\mathbf{y}_u(u, \sin v))\\ &= 0.\\ (F^*\phi_2)(\mathbf{x}_u(u, v)) &= \phi_2(F_*(\mathbf{x}_u(u, v)))\\ &= \phi_2(\mathbf{y}_u(u, \sin v))\\ &= 0.\\ (F^*\phi_2)(\mathbf{x}_v(u, v)) &= \phi_2(F_*(\mathbf{x}_v(u, v)))\\ &= \phi_2(\cos v\, \mathbf{y}_v(u, \sin v))\\ &= \cos v\, \phi_2(\mathbf{y}_v(u, \sin v))\\ &= \cos v \times 1\\ &= \cos v.\end{aligned}$$

(b) By Theorem 5.7,

$$F^*(\phi_1 \wedge \phi_2) = (F^*\phi_1) \wedge (F^*\phi_2).$$

If we use this and the definition of wedge product, we obtain

$$\begin{aligned}F^*(\phi_1 \wedge \phi_2)(\mathbf{x}_u, \mathbf{x}_v) &= (F^*\phi_1)(\mathbf{x}_u) \times (F^*\phi_2)(\mathbf{x}_v)\\ &\quad - (F^*\phi_1)(\mathbf{x}_v) \times (F^*\phi_2)(\mathbf{x}_u)\\ &= 1 \times \cos v - 0 \times 0\\ &= \cos v.\end{aligned}$$

Note: We have dropped the parameters wherever it seemed safe to do so. This is usually where there is no doubt as to their value. When there are several sets of parameter values around, we retain them in all expressions.

Solution 5.2

This solution is an example of where it is a good idea to retain parameters because there are two sets around:

$$(u, v)$$

and

$$(u + v, u - v).$$

We start by calculating an explicit expression for U. As usual, the first things that we need are the partial velocities.

$$\mathbf{x}_u(u, v) = (1, 0, v),$$
$$\mathbf{x}_v(u, v) = (0, 1, u).$$

It follows that

$$\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v) = (-v, -u, 1),$$
$$\|\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v)\| = \sqrt{1 + u^2 + v^2}.$$

Thus

$$U(\mathbf{x}(u, v)) = \frac{(-v, -u, 1)}{\sqrt{1 + u^2 + v^2}}.$$

We now apply the definition of η.

$$\begin{aligned}\eta(\mathbf{x}_u(u, v), \mathbf{x}_v(u, v)) &= \mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v) \cdot U(\mathbf{x}(u, v))\\ &= (-v, -u, 1) \cdot U(\mathbf{x}(u, v))\\ &= \frac{1 + u^2 + v^2}{\sqrt{1 + u^2 + v^2}}\\ &= \sqrt{1 + u^2 + v^2}.\end{aligned}$$

In order to find $F^*\eta$, we shall need $F_*(\mathbf{x}_u)$ and $F_*(\mathbf{x}_v)$. Using partial differentiation and the chain rule, we obtain the following.

$$\begin{aligned}F_*(\mathbf{x}_u(u, v)) &= \frac{\partial}{\partial u}F(\mathbf{x}(u, v))\\ &= \frac{\partial}{\partial u}\mathbf{x}(u + v, u - v)\\ &= \mathbf{x}_u(u + v, u - v)\frac{\partial(u + v)}{\partial u}\\ &\quad + \mathbf{x}_v(u + v, u - v)\frac{\partial(u - v)}{\partial u}\\ &= \mathbf{x}_u(u + v, u - v) + \mathbf{x}_v(u + v, u - v).\\ F_*(\mathbf{x}_v(u, v)) &= \frac{\partial}{\partial v}F(\mathbf{x}(u, v))\\ &= \frac{\partial}{\partial v}\mathbf{x}(u + v, u - v)\\ &= \mathbf{x}_u(u + v, u - v)\frac{\partial(u + v)}{\partial v}\\ &\quad + \mathbf{x}_v(u + v, u - v)\frac{\partial(u - v)}{\partial v}\\ &= \mathbf{x}_u(u + v, u - v) - \mathbf{x}_v(u + v, u - v).\end{aligned}$$

To complete the solution we use the general result

$$\eta(a\mathbf{x}_u + b\mathbf{x}_v, c\mathbf{x}_u + d\mathbf{x}_v) = (ad - bc)\eta(\mathbf{x}_u, \mathbf{x}_v).$$

We also observe that the expression obtained for η means

$$\eta(\mathbf{x}_u, \mathbf{x}_v) = \sqrt{1 + (\text{1st parameter})^2 + (\text{2nd parameter})^2}.$$

Since

$$(F^*\eta)(\mathbf{x}_u(u, v), \mathbf{x}_v(u, v)) = \eta(F_*(\mathbf{x}_u(u, v)), F_*(\mathbf{x}_v(u, v)))$$

and we already have expressions for

$$F_*(\mathbf{x}_u) \text{ and } F_*(\mathbf{x}_v),$$

we can put everything together to obtain as follows.

$$\begin{aligned}
&(F^*\eta)(\mathbf{x}_u(u,v),\mathbf{x}_v(u,v))\\
&= \eta(F_*(\mathbf{x}_u(u,v)), F_*(\mathbf{x}_v(u,v)))\\
&= \eta(\mathbf{x}_u+\mathbf{x}_v, \mathbf{x}_u-\mathbf{x}_v)\\
&\quad \text{(evaluated at } (u+v, u-v))\\
&= (1\times(-1) - 1\times 1)\eta(\mathbf{x}_u(u+v,u-v), \mathbf{x}_v(u+v,u-v))\\
&= -2\eta(\mathbf{x}_u(u+v,u-v), \mathbf{x}_v(u+v,u-v))\\
&= -2\sqrt{1+(u+v)^2+(u-v)^2}\\
&= -2\sqrt{1+2u^2+2v^2}.
\end{aligned}$$

Note that we have used the result

$$\eta(a\mathbf{x}_u + b\mathbf{x}_v, c\mathbf{x}_u + d\mathbf{x}_v) = (ad-bc)\eta(\mathbf{x}_u,\mathbf{x}_v).$$

Solution 6.1

In the long run it is probably easiest to give an explicit expression for $\mathbf{x}$.

$$\mathbf{x} = ((1+v\cos(u/2))\cos u, (1+v\cos(u/2))\sin u, v\sin(u/2)).$$

(a) The partial velocities are

$$\begin{aligned}
\mathbf{x}_u &= (-(1+v\cos(u/2))\sin u - (v/2)\sin(u/2)\cos u,\\
&\quad (1+v\cos(u/2))\cos u - (v/2)\sin(u/2)\sin u,\\
&\quad (v/2)\cos(u/2));\\
\mathbf{x}_v &= (\cos(u/2)\cos u, \cos(u/2)\sin u, \sin(u/2)).
\end{aligned}$$

The easiest check for regularity is probably that given by calculating the expression $EG-F^2$. Careful algebra yields

$$\begin{aligned}
E &= \mathbf{x}_u\cdot\mathbf{x}_u\\
&= (1+v\cos(u/2))^2(\cos^2 u+\sin^2 u) + (v^2/4)\\
&= (1+v\cos(u/2))^2 + (v^2/4);\\
F &= \mathbf{x}_u\cdot\mathbf{x}_v\\
&= 0;\\
G &= \mathbf{x}_v\cdot\mathbf{x}_v\\
&= \cos^2(u/2)(\cos^2 u+\sin^2 u) + \sin^2(u/2)\\
&= 1.
\end{aligned}$$

Since $E\geq 1$, it follows that $EG-F^2\neq 0$ and $\mathbf{x}$ is regular.

(b) We need a value of u_0 such that

$$\cos u_0 = 1, \ \sin u_0 = 0.$$

The easiest non-zero value is $u_0 = 2\pi$.

We have

$$\begin{aligned}
\mathbf{x}(0,0) = \mathbf{x}(2\pi,0) &= (1,0,0);\\
\mathbf{x}_u(0,0) &= (0,1,0),\\
\mathbf{x}_u(2\pi,0) &= (0,1,0),\\
\mathbf{x}_v(0,0) &= (1,0,0),\\
\mathbf{x}_v(2\pi,0) &= (-1,0,0).
\end{aligned}$$

It follows that

$$\mathbf{x}_u(0,0)\times\mathbf{x}_v(0,0) = -\mathbf{x}_u(2\pi,0)\times\mathbf{x}_v(2\pi,0).$$

(c) All the coordinate functions that appear in the definition of f are differentiable and are combined in

$$f(\mathbf{x}(u,v)) = \mathbf{x}_u(u,v)\times\mathbf{x}_v(u,v)\cdot Z(\mathbf{x}(u,v))$$

using only multiplication and addition. It follows that f is differentiable and hence continuous.

(d) We can regard $f(\mathbf{x}(u,0))$ as a continuous function $g(u)$. However,

$$\begin{aligned}
g(0) &= f(\mathbf{x}(0,0))\\
&= \mathbf{x}_u(0,0)\times\mathbf{x}_v(0,0)\cdot Z(\mathbf{x}(0,0))
\end{aligned}$$

and

$$\begin{aligned}
g(2\pi) &= f(\mathbf{x}(2\pi,0))\\
&= \mathbf{x}_u(2\pi,0)\times\mathbf{x}_v(2\pi,0)\cdot Z(\mathbf{x}(2\pi,0))\\
&= -\mathbf{x}_u(0,0)\times\mathbf{x}_v(0,0)\cdot Z(\mathbf{x}(0,0)).
\end{aligned}$$

The last line follows because $\mathbf{x}(0,0)=\mathbf{x}(2\pi,0)$ and just $\mathbf{x}_v$ changes sign.

It follows that, because g changes sign, it must vanish on $v=0$.

However, g is defined as a dot product. The first term is

$$\mathbf{x}_u\times\mathbf{x}_v\neq 0,$$

because $\mathbf{x}$ is regular. It follows that, since

$$\mathbf{x}_u\times\mathbf{x}_v$$

is normal to the surface (and cannot be perpendicular to Z), that Z must vanish. This contradiction shows that M is not orientable because any attempt to define a non-vanishing normal vector field on M must fail.

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